Polynomial Methods for Control Analysis and Design

[PolyX]

1/ Polynomials and polynomial matrices

Overview

[PolyX]

Ch. 1. Polynomials and polynomial matrices

Ch. 2. Polynomial toolbox — computer session

Ch. 3. Polynomials in control systems

Ch. 4. Discrete-time systems

Ch. 5. Continuous-time and MIMO systems

Ch. 6. CAD based on polynomial methods — computer session

Ch. 7. Future perspectives

Overview

Ch. 2. Polynomials and polynomial matrices

Preview: polynomials in control

Transfer function and matrix
Feedback loop
Simple analysis
Simple synthesis
Polynomial equation

Transfer function and transfer matrix

SISO systems

Often described by rational transfer functions looking like

$$\boxed{\frac{b(s)}{a(s)}}$$

But this is nothing else than a polynomial fraction.

MIMO systems

Often described by rational transfer matrix looking like

$$\boxed{\begin{bmatrix} \frac{b_{11}(s)}{a_{11}(s)} & \cdots \\ \cdots & \ddots \end{bmatrix}}$$

But this is nothing else than a matrix of polynomial fractions.

Feedback loop

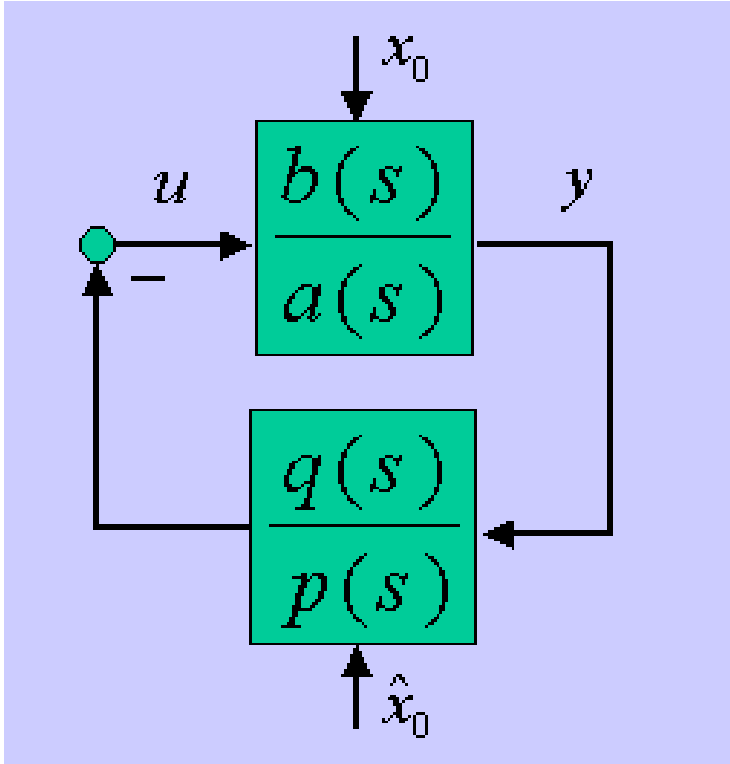
If there are no hidden modes in the plant and controller descriptions, then

$$\boxed{a(s)p(s)+b(s)q(s)}$$

is the

characteristic polynomial

of the closed loop



Analysis

Task A: Analysis

Given plant $a(s), b(s)$ and controller $p(s), q(s)$,

compute $a(s)p(s) + b(s)q(s)$ and analyze it.

May check

- stability
- position of poles
- robustness
- ....

Synthesis

Task B: Synthesis

Desired c-l characteristic polynomial

Given plant $a(s), b(s)$ and a polynomial $c(s)$,

compute a controller $p(s), q(s)$ so that

$$a(s)p(s) + b(s)q(s) = c(s)$$

The resulting controller guarantees that the closed-loop characteristic polynomial is $c(s)$ and hence has the desired properties!

Polynomial equation

[PolyX]

Basic problem:

Given $a(s), b(s)$ and $c(s)$, how one gets $p(s), q(s)$

such that $a(s)p(s) + b(s)q(s) = c(s)$???

It must be solved as an equation in polynomials !

This is a LINEAR POLYNOMIAL EQUATION

Polynomial

Various representations

Polynomial by coefficients

Polynomial represented by coefficients

coefficients

degree

$$p(s) = p_0 + p_1 s + \cdots + p_n s^n$$

constant coefficient

variable, indeterminate

leading coefficient

$\deg p(s) = n$
$\Leftrightarrow p_n \neq 0;$
$\deg 0 = -\infty$

the representation used in the Polynomial Toolbox

Polynomial by values

Polynomial represented by values

$n+1$ distinct complex points

$$p(s_0), p(s_1), p(s_2), \ldots, p(s_n) \in C$$

$n+1$ complex values

Polynomial by roots

Polynomial represented by its roots

- if complex numbers allowed

n complex roots

$$p(s) = c(s - s_1)\cdots(s - s_n)$$

- if complex numbers are not allowed, then represent each complex conjugate pair by the second degree polynomial

$$s^2 - 2\alpha s + \alpha^2 + \beta^2 = (s - (\alpha + i\beta))(s - (\alpha - i\beta))$$

Polynomial Toolbox 2

Key features
Local installation
Initialization
poldesk
Manual

Polynomial Toolbox

Polynomial Toolbox 2.0

[PolyX]

MATLAB PARTNER

- Based on Matlab v. 5 or later
- Object oriented:
  polynomials and polynomial matrices
  are defined as objects
- More than 200 routines
- Stand-alone but easily cooperates with
  Simulink, Control Systems Toolbox,
  Symbolic Math Toolbox

Key Features

- Simple input, manipulation and display of polynomials and polynomial matrices based on a new polynomial matrix object
- Overloaded operations and functions, solvers for numerous linear and quadratic matrix polynomial equations
- Polynomial matrices with complex coefficients for applications in signal processing
- New generation of numerical algorithms: easy, fast, reliable
- Polynomial Matrix Editor, 2-D and 3-D color plots

Key features - 2

- Continuous-time and discrete-time system and signal models based on polynomial matrix fractions
- Classical and robustness analysis for LTI systems and filters
- Classical and optimal design tools: pole placement, all stabilizing controllers, dead-beat, H2 and LQG
- H-infinity optimization in a generality not found elsewhere
- Robust control with parametric uncertainties: single parameter, interval and poly-topic

Key features - 3

- Conversion to and from LTI objects of the Control System Toolbox and polynomial objects defined in the Symbolic Math Toolbox

- Simulink block set for LTI systems described by polynomial matrix fractions

Local installation

- ?

Initialization

Initialization command

```
» pinit

  Polynomial Toolbox initialized. To get started, type one of
  these: helpwin or poldesk. For product information, visit
  www.polyx.com or www.polyx.cz.
```

On line help: in pdf

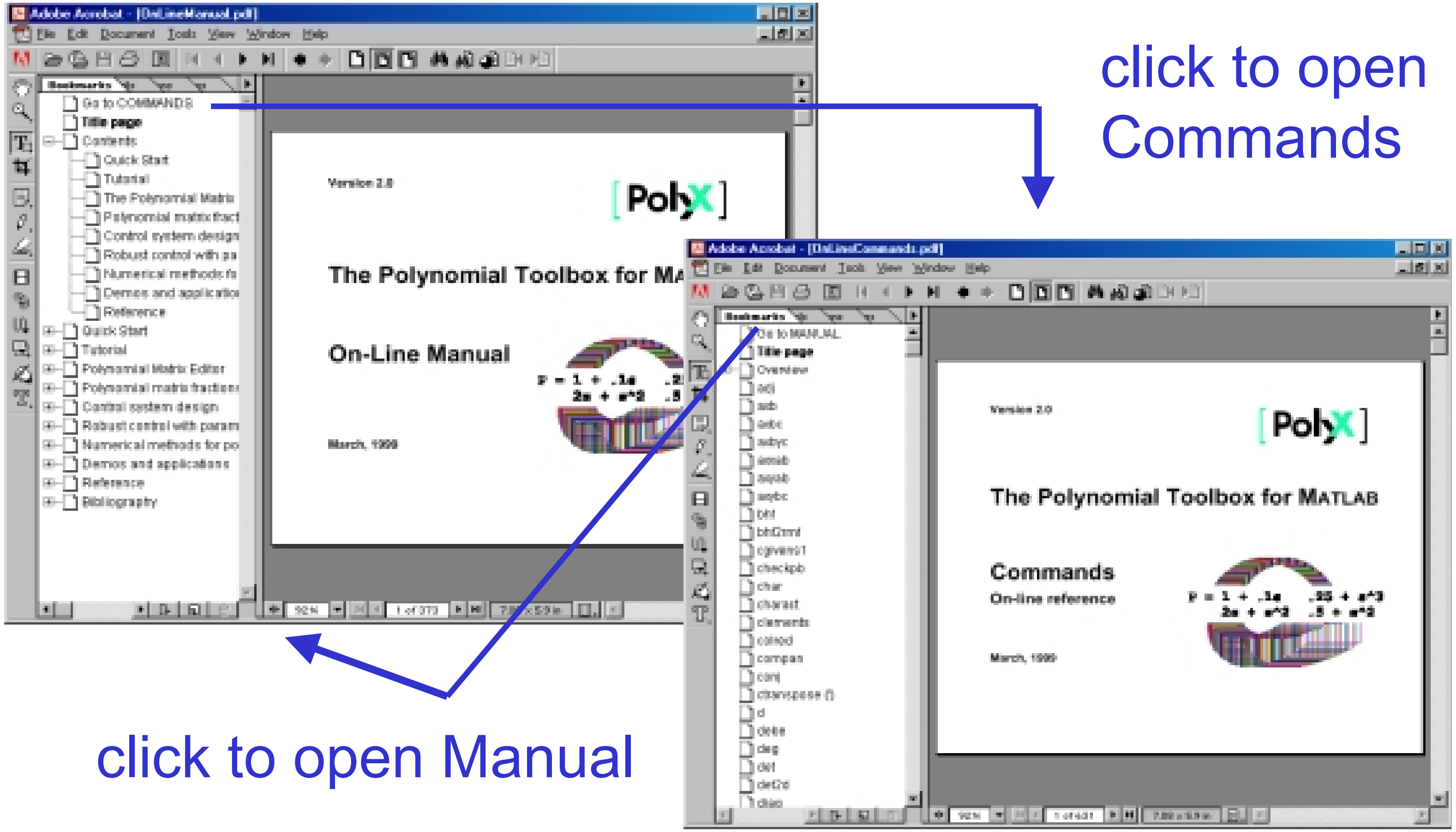
PolyX Webs

On line help: in Matlab help window

poldesk

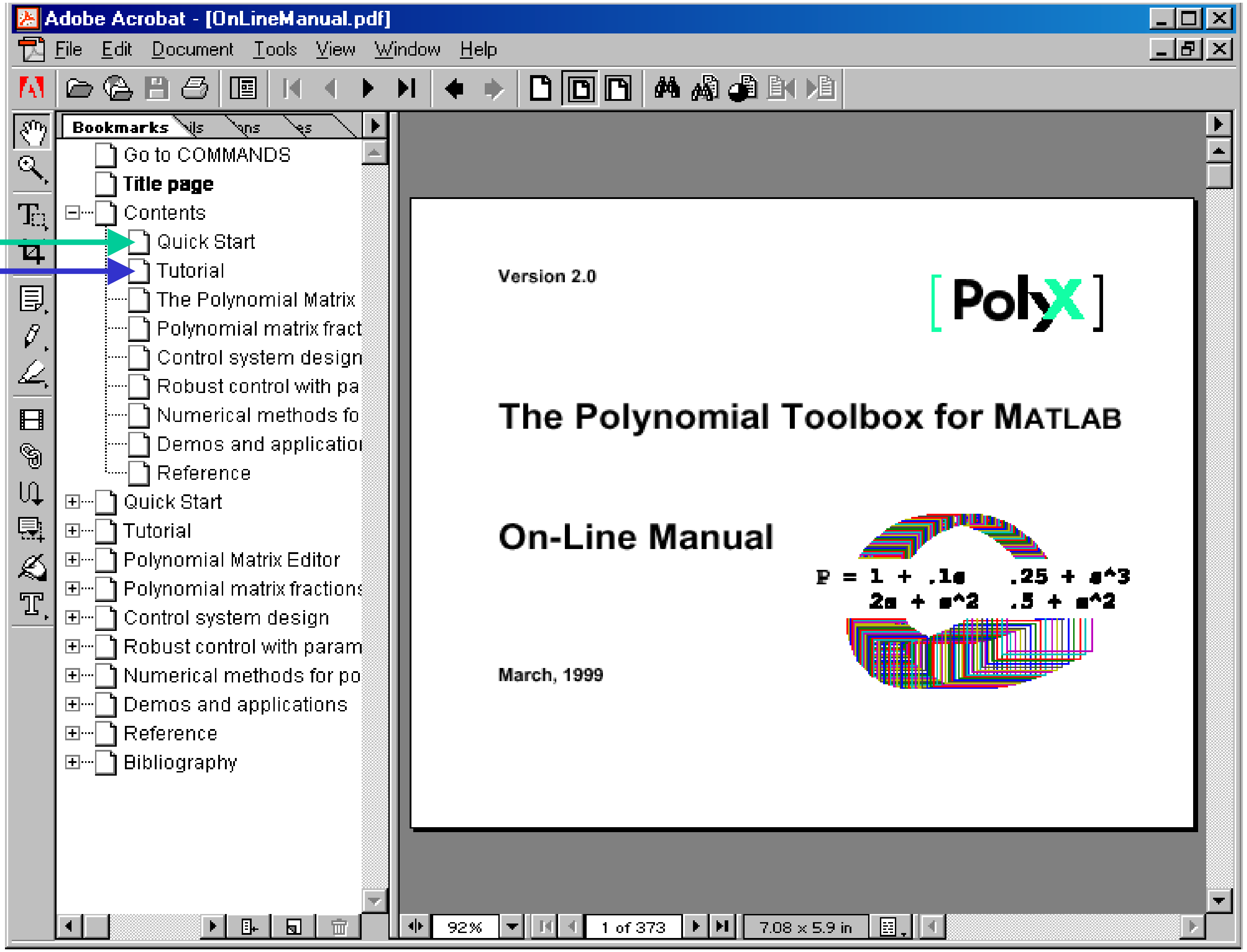
» poldesk

click to open Commands



click to open Manual

Manual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Version 2.0
[PolyX]
The Polynomial Toolbox for MATLAB
On-Line Manual
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Rings and fields

Rings
Groups
Commutative rings
Units and fields
Divisors and multiples
Primness and coprimness

Rings

Ring

Set G equipped with two operations

- addition
- multiplication

which is

- a commutative group with respect to addition
- a semigroup with respect to multiplication

such that

- multiplication distributes over addition

Groups

Group with respect to addition

- the addition is associative
- the addition is commutative
- a neutral element 0 exists
- the additive inverse exists

Semigroup with respect to multiplication

- multiplication is associative
- multiplicative neutral element 1 exists

Commutative rings

Commutative ring

Ring such that multiplication is commutative

Examples of commutative rings

- the set of integers
- the set of all polynomials

Units and fields

If an element u which belongs to a ring has a multiplicative inverse then u is called a unit

Examples:

- The set of integers has the units +1 and –1
- Units of the set of polynomials are nonzero real numbers

Field

Commutative ring in which the zero and unit element are different and each nonzero element is a unit

Examples:

- The rationals, the reals, the complex numbers, the rational functions

Divisors and multiples

Notions for elements of a commutative ring

- a is a divisor of b if there exists a c such that $b = ac$
- a is a multiple of b if there exists a c such that $a = bc$
- If g divides both a and b then it is a common divisor of a and b
- If g is a multiple of every common divisor of a and b then it is a greatest common divisor
- If l is a multiple of both a and b then it is called a common multiple of a and b
- If l is a divisor of every common multiple of a and b then it is a least common multiple of a and b

Coprimness and primness

- a and b are coprime if their only common divisors are units of the ring

- a is prime if it is divisible only by units and elements of the form au, with u a unit

Ring of polynomials

Features
Euclidean division
GCD and LCM
Roots
Stability

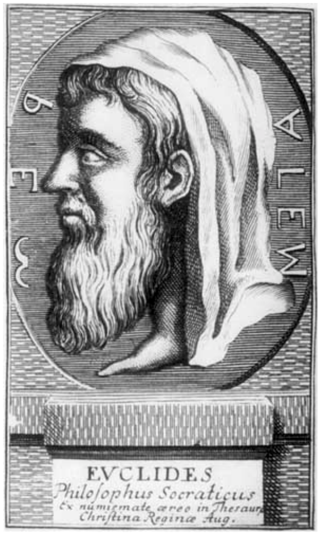
Features

Ring of polynomials in s: $\boxed{R[s],\ C[s]}$

($R(s)$, $C(s)$ are sequences or fractions)

- units = polynomials of degree 0 (isomorphic with nonzero constants)
- other polynomials are not units so it is only a ring not a field!
- it has no zero divisors (no $a,b\neq 0: ab=0$) so it is an integral domain
- prime elements = irreducible polynomials in $R[s]$ of the form $p_0+p_1 s$ or $p_0+p_1 s+p_2 s^2,\ p_1^2-4p_0 p_2<0$

Euclidean division



Euclid (Alexandria, ~ 300 BC)

- basic tool for studying divisibility

Euclidean ring

Given polynomials a, b with $b \neq 0$
polynomials q, r exist
such that

$$a = bq + r, \;\; \deg r < \deg b$$

quotient — remainder

GCD and LCM

[PolyX]

Formulas for GCD and LCM

For any polynomials $a, b \in C[s]$ we can always find their gcd g and lcm l along with two pairs of coprime polynomials p, q and v, w such that

$$ap + bq = g$$

$$av + bw = 0$$

and

$$l = av = -bw$$

PID
Principal Ideal Domain

(not true for polynomials in two variables)

Roots

[PolyX]

Definition:

Polynomial $p(s)$ has a root $s_i \in \mathrm{C}$ if $p(s_i) = 0$

Facts:

- **Fundamental Theorem of Algebra**

  A polynomial of degree n

  $$p(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n, \quad p_n \neq 0$$

  has exactly n roots (when counted with multiplicities)

- **Complex conjugate pairs**

  If all the coefficients p_i are real, then complex roots come in complex conjugate pairs $(\alpha + i\beta, \alpha - i\beta)$

Stability

Location of roots

- plays an important role in technical applications of polynomials
- reveals stability and other dynamical properties of underlying systems

A polynomial is

- Hurwitz stable iff all its roots are in open left half plane
- Schur stable iff all its roots are inside of unit disc
- Stable in z^{-1} or d - " - outside - " –
- *D*-stable iff all its roots are in *D*

Notation and formulas

$a \mid b$ stands for a divides b

$\gcd(a,b) = (a,b)$ two options denoting gcd

$$\deg a(s)b(s) = \deg a(s) + \deg b(s)$$

Polynomial fractions

Field of fractions
Properness
Causality
Poles and zeros
Stability

Field of fractions

Consider the ring of polynomials $\mathsf{R}[s]$ and fractions of the form

$$a = \frac{a_2}{a_1}, \;\; a_1, a_2 \in \mathsf{R}[s], a_1 \neq 0$$

$$b = \frac{b_2}{b_1}, \;\; b_1, b_2 \in \mathsf{R}[s], b_1 \neq 0$$

With equality $a = b$ defined by $a_2 b_1 = b_2 a_1$ and addition and multiplication by

$$a + b = \frac{a_2 b_1 + a_1 b_2}{a_1 b_1}$$

$$ab = \frac{a_2 b_2}{a_1 b_1}$$

the set of equivalence classes of these fractions is a field. Each class can be characterized by a fraction with coprime numerator and denominator

Properness

Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

is

- proper if $$\deg p(s) \geq \deg q(s)$$
- strictly proper if $$\deg p(s) > \deg q(s)$$
- improper if $$\deg p(s) < \deg q(s)$$
- biproper if $$\deg p(s) = \deg q(s)$$

Causality

Polynomial fraction

$$h(d) = \frac{b(d)}{a(d)}$$

is

- causal if $a(0) \neq 0$
- strictly causal if $a(0) \neq 0, b(0) = 0$
- improper if $a(0) = 0, b(0) \neq 0$
- biproper if $a(0) \neq 0, b(0) \neq 0$

Poles and zeros

Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

- Poles: Roots of p
- Zeros: Roots of q

Poles and zeros - 2

Definition:

- Poles/zeros at Infinity
  Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

has a pole/zero at infinity iff the polynomial fraction

$$h(1/s)$$

has a pole/zero at 0.

Poles and zeros - 3

Facts: about $h(s) = \dfrac{q(s)}{p(s)}$

- If $\deg p(s) > \deg q(s)$ then it has at infinity a zero with multiplicity $k = \deg p(s) - \deg q(s)$

- If $\deg p(s) < \deg q(s)$ then it has at infinity a zero with multiplicity $l = \deg q(s) - \deg p(s)$

- When taking poles/zeros at infinity into account, then

  Number of poles = Number of zeros

Stability

Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

- is stable iff its denominator p is a stable polynomial

Polynomial equations

Polynomial equation
Solvability
General solution
Minimum degree solution
Coincidence
Some algorithms

Preview

Equations in a field look like this

$$ax=b$$

and are solved by like this

$$x=a\backslash b$$

This is hardly solvable in a field !

(unless a is a unit)

More useful is

$$ax+by=c$$

Polynomial equations

[PolyX]

Linear polynomial equations - also called Diophantine

(in integers by Diophantus from Alexandria, ~300 AD)

no picture found

Given polynomials

$$a(s)x(s) + b(s)y(s) = c(s)$$

Polynomials to be computed

Solvability

Solvability condition

The equation

$$a(s)x(s)+b(s)y(s)=c(s)$$

is solvable if and only if

$$\gcd(a,b)\,|\,c$$

(the greatest common divisor of a and b divides c)

Proof

Only if:

Let $ax' + by' = c$

and write $(a,b) = g,\ a = g\bar{a},\ b = g\bar{b}$

Then $g(\bar{a}x' + \bar{b}y') = c$ and hence $g \mid c$ •

If:

Let $(a,b) \mid c$

and denote $(a,b) = g,\ c = g\bar{c}$

Then there always exist p,q such that $ap + bq = g$

Multiplying by $\bar{c}$ we get $a(p\bar{c}) + b(q\bar{c}) = c$

and hence he have constructed a solution

$$x = p\bar{c},\ y = q\bar{c}$$ •

General solution

Denoting $$\bar{a} = \frac{a}{\gcd(a,b)}, \quad \bar{b} = \frac{b}{\gcd(a,b)}$$

Note that "-" can be "everywhere"

the general solution reads

solutions are infinitely many!

$$\begin{aligned} x(s) &= x'(s) - \bar{b}(s)t(s) \\ y(s) &= y'(s) + \bar{a}(s)t(s) \end{aligned}$$

particular solution

arbitrary polynomial parameter

Proof

Proof:

By assumption $ax' + by' = c$

and hence $a(x - x') + b(y - y') = 0$.

Now polynomials $\bar{a}, \bar{b}$ defined before are coprime and satisfy $\bar{a}b = a\bar{b}$.

As a result $\bar{b} \mid x - x'$ and $\bar{a} \mid y - y'$, that is,

$$x - x' = -\bar{b}t$$

$$y - y' = \bar{a}t$$

for some polynomial t . To obtain any solution, we let t range over $\mathrm{R}[s]$ and the claim follows. •

Minimum degree solutions

Take the general solution and use the division algorithm to reduce

$$x = x' - \bar{b}t$$
$$y = y' + \bar{a}t$$

x' modulo $\bar{b}$: $x' = \bar{b}q + r, \quad \deg r < \deg \bar{b}$

Then $x = r - \bar{b}(t - q)$

and the minimum degree solution x, y wrt x becomes

$$\boxed{\begin{aligned} x &= r \\ y &= y' + \bar{a}q \end{aligned}}$$

$$\boxed{\deg x < \deg \bar{b}}$$

either is unique

There is a(other) minimum degree solution x, y wrt y characterized by $\boxed{\deg y < \deg \bar{a}}$

but may not be identical

Coincidence condition

An important condition

$$(a,b)=1: \;\; ax+by=c \;\mapsto\; \boxed{\frac{x}{b}}+\boxed{\frac{y}{a}}=\frac{c}{ab}$$

s.p. if sol. is min.deg.wrt x

s.p. if sol. is min.deg.wrt y

s.p. if

$\deg c < \deg a + \deg b$

If RHS s.p. then either both LHE are s.p. or none!

If RHS is not s.p., then only one of LHE can be s.p. at a time!

Coincidence

One minimum degree solution

If $\deg c < \deg a + \deg b$

then both minimum degree solutions coincide

and hence there exists only one min. deg. sol.

(which has min. degree of both x and y!)

Otherwise, if $\deg c \geq \deg a + \deg b$

then there are really two different minimum degree solutions.

Some algorithms

How can you solve polynomial equations?

Use the Polynomial Toolbox !

Only if unplugged, try (for simple examples) your pencil (and back side of an envelope mentioned by Mrs. A. Einstein) and one of these routines:

- Solution via elementary operations (reductions)
- Solution using Sylvester matrix

to be described.

Other methods not explained here include

- State-space solution or
- Interpolation

.

Polynomial reductions

Elementary operations on a polynomial matrix

Row operations: 3 basic

- multiplying a row by a nonzero constant

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{multiply first row by 3}} \begin{bmatrix} 3 & 3s \\ 2 & s^2 \end{bmatrix}$$

- interchanging two rows

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{interchange the first and second row}} \begin{bmatrix} 2 & s^2 \\ 1 & s \end{bmatrix}$$

Polynomial reductions - 2

- adding a polynomial multiple of one row to another

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow[\text{and add the result to the first row}]{\text{multiply the second row by } s} \begin{bmatrix} 1+2s & s+s^3 \\ 2 & s^2 \end{bmatrix}$$

Column operations are dual

(Elementary operations preserve determinant
= multiplication by a unimodular matrix)

Solution via polynomial reductions

Solution of $a(s)x(s)+b(s)y(s)=c(s)$ via reductions

- Step 1 Form a composite matrix

$$\begin{bmatrix} a(s) & 1 & 0 \\ b(s) & 0 & 1 \end{bmatrix}$$

- Step 2 Reduce it via elementary row ops.

$$\begin{bmatrix} g(s) & p(s) & v(s) \\ 0 & q(s) & w(s) \end{bmatrix}$$

Solution via polynomial reductions - 2

Then $$\begin{aligned} p(s)a(s)+q(s)b(s) &= g(s) \quad \text{gcd}(a,b) \\ v(s)a(s)+w(s)b(s) &= 0 \end{aligned}$$

- Step 3 Extract $g(s)$ from $c(s)$ to get

$$c(s)=\bar{c}(s)g(s)$$

If not possible → **stop!**

- Step 4 Take

$$\boxed{\begin{aligned} x(s)&=\bar{c}(s)p(s) \\ y(s)&=\bar{c}(s)q(s) \end{aligned}}$$

Solution via polynomial reductions - 3

- Moreover, all solutions are expressed as

$$x(s) = \bar{c}(s)p(s) + v(s)t(s)$$
$$y(s) = \bar{c}(s)q(s) + w(s)t(s)$$

free
polynomial
parameter

Solution via Sylvester matrix - 1

Solution of $a(s)x(s)+b(s)y(s)=c(s)$ via reductions

The method will be explained for a simple case with given

$$\begin{aligned} a(s) &= a_0 + a_1 s + a_2 s^2 \\ b(s) &= b_0 + b_1 s + b_2 s^2 \\ c(s) &= c_0 + c_1 s + c_2 s^2 \end{aligned}$$

and expected

$$\begin{aligned} x(s) &= x_0 + x_1 s \\ y(s) &= y_0 + y_1 s \end{aligned}$$

Solution via Sylvester matrix - 2

[PolyX]

- Step 1 Expand and equate coefficients at like powers
  to get a linear equation with constant matrices

$$\begin{bmatrix} x_0 & y_0 & x_1 & y_1 \end{bmatrix}\begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{bmatrix}=\begin{bmatrix} c_0 & c_1 & c_2 & 0 \end{bmatrix}$$

unknowns

Sylvester, resultant matrix

Solution via Sylvester matrix - 3

- Step 2 Solve the Sylvester equation

$$\Rightarrow \qquad x_0, x_1, y_0, x_0$$

- Step 3 Construct the polynomials

$$\boxed{\begin{aligned} x(s) &= x_0 + x_1 s \\ y(s) &= y_0 + y_1 s \end{aligned}}$$

Bezout equation

Bezout equation

- looks like this $$a(s)x(s)+b(s)y(s)=1$$
- is a particular case of Diophantine equation
- is solvable iff a and b are coprime
- has unique minimum degree solution as coincidence is guaranteed

Open problems

The Diophantine equations

- have been intensively studied: over decades in the polynomial case, or even over millennia in the integer case
- so they are well understood yet,
- surprisingly, some problems are still open

Open problems are e.g.

- What is real degree of the min. deg. sol.? The condition $\deg x < \deg \bar{b}$ generically reads $\deg x = \deg \bar{b} - 1$ but may be $\deg x < \deg \bar{b} - 1$ When and why?
- When the solution fails to be coprime?
- When the solution satisfies additional conditions (e.g. x stable) and how to find it?

Polynomial matrices

Polynomial matrices
Divisibility
Coprimness and primness
GCD and LCM
Division with reminder
Zeros

Polynomial matrices

Polynomial matrices are matrices over the ring of polynomials.

There are two ways how to look at a polynomial matrix $P(s)$

1) P is a matrix whose entries are polynomials

$$P(s) = \begin{bmatrix} 1+s & s^2 \\ 3+s^3 & 4 \end{bmatrix}$$

2) P is a polynomial whose coefficients are matrices

$$P(s) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s^3$$

Algebraic structure

- Square polynomial matrices $m \times m,\ m > 1$ form a noncommutative ring
- A square matrix $A(s)$ is a unit in the ring iff $\det A(s)$ is a unit in the ring of scalar polynomials, that is, iff $\det A(s)$ is a nonzero constant. Such matrix is called unimodular.
- A square matrix $A(s)$ is a zero divisor iff $\det A(s) = 0$

Divisibility

Consider polynomial matrices *A, B* and *C.* If *A=BC*, then

- *B* is a left divisor of *A*
- *A* is a right multiple of *B*
- *C* is a right divisor of *A*
- *A* is a left multiple of *C*
- If *B* is square unimodular, then *A* and *C* are left equivalent
- If *C* is square unimodular, then *A* and *B* are right equivalent
- If *A=BCD* with both *B* and *D* square unimodular, then *A* and *C* are equivalent.

Consider now polynomial matrices *A* and *B* with the same number of rows (columns):

- If G_1 is a left (right) divisor of both *A* and *B,* then it is their common left (right) divisor
- If, furthermore, G_1 is a right (left) multiple of every common left (right) divisor of *A* and *B,* then it is their greatest common left (right) divisor.

Coprimness

- Matrices *A,B* with the same number of rows (columns) are left (right) coprime if their only common left (right) divisors are unimodular matrices.
- A matrix *A* is left (right) prime if their only left divisors are unimodular matrices.
- Matrices *A,B* are left coprime if the composite matrix [*A,B*] is left prime and vice versa.
- Matrices *A,B* are right coprime if the composite matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is right prime and vice versa.
- Such a way, coprimness can easily be generalized for 3 and more matrices.

GCD and LCM – matrix case

Formulas for matrix GCD and LCM

- For any two polynomial matrices A,B with the same number of rows, we can always find their gcld G_1 and lcrm L_1 along with two pairs of right coprime matrices P_1, Q_1 and V_1, W_1 such that

$$\begin{aligned} AP_1 + BQ_1 &= G_1 \\ AV_1 + BW_1 &= 0 \end{aligned}$$

and

$$L_1 = AV_1 = -BW_1$$

- For any two polynomial matrices A,B with the same number of columns, we can always find their gcrd G_2 and lclm L_2 along with two pairs of right coprime matrices P_2, Q_2 and V_2, W_2 such that

$$\begin{aligned} P_2A + Q_2B &= G_2 \\ V_2A + W_2B &= 0 \end{aligned}$$

and

$$L_2 = V_2A = -W_2B$$

Division with a remainder for matrices

PolyX

Left division:

- Consider two matrices *N*,*D* with the same number of rows, *D* square and full rank. Then there exist unique polynomial matrices *Q*,*R* such that

  $N = DQ + R$ and $D^{-1}R$ is a strictly proper rational matrix

- This division corresponds to splitting of a rational matrix into a polynomial and strictly proper part

$$\underbrace{D^{-1}N}_{\text{rational}} = \underbrace{Q}_{\text{polynomial part}} + \underbrace{N^{-1}R}_{\text{strictly proper part}}$$

- Right division with a reminder is defined dually.

Zeros

Zero

- of a polynomial matrix *P* is defined as a complex number s_i such that

$$\operatorname{rank} P(s_i) < \operatorname{rank} P(s)$$

- If *P* is square full rank, then its zeros equal roots of

$$\det P(s)$$

Polynomial matrix fractions

Left and right fractions
Conversion rational-pmf
Conversion lmf-rmf

Polynomial matrix fractions

How can we generalize 'polynomial fraction' for matrices?

- A rational matrix F can be described by means of two polynomial matrices as

left fraction

$$F(s) = D(s)^{-1} N(s)$$

polynomial matrix numerator

left polynomial matrix denominator
(square, invertible)

fraction is coprime if D,N are so

- Alternatively, it can be described by means of other two polynomial matrices as

right fraction

$$F(s) = N(s) D(s)^{-1}$$

right polynomial matrix denominator
(square, invertible)

polynomial matrix numerator

Conversions: rational and PMF

Rational to left PMF:

1) Take least common denominator in each row of F
2) Write $F(s) = diag\{1/d_i(s)\}N(s) = \underbrace{diag\{d_i(s)\}}_{D(s)}{}^{-1} N(s)$
3) Make it coprime if required.

Left PMF to rational:

$$D^{-1}(s)N(s) = \frac{1}{\det D(s)} \overbrace{(\operatorname{adj} D(s))\, N(s)}^{F_N(s)} = \left[\frac{f_{N,ij}}{\det D} \right]$$

and make each entry coprime if required.

For Right PMF similarly.

Conversions: left-right

Left to right PMF

Given a left PMF $D^{-1}(s)N(s)$, compute right PMF as follows:

- Make a composite matrix $[D(s), N(s)]$
- Solve the equation (find a right null space, in fact)

$$[D(s), N(s)]\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = 0$$

- Then the desired right fraction reads $Y(s)X^{-1}(s)$

Right to left PMF

Given a right PMF $N(s)D^{-1}(s)s$

- Make a composite matrix $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ and solve $[X(s), Y(s)]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$
- Then the desired right fraction reads $X^{-1}(s)Y(s)$

Polynomial matrix equations

axbyc
xaybc
axybc

Matrix equations in polynomials

A natural generalization of

$$a(s)x(s)+b(s)q(s)=c(s)$$

is either

$$A(s)X(s)+B(s)Y(s)=C(s)$$

axbyc

where $A \in \mathbb{R}_{lp}[s], B \in \mathbb{R}_{lq}[s], C \in \mathbb{R}_{lm}[s]$

are given polynomial matrices or

$$X(s)A(s)+Y(s)B(s)=C(s)$$

xaybc

where $A \in \mathbb{R}_{pm}[s], B \in \mathbb{R}_{qm}[s], C \in \mathbb{R}_{lm}[s]$

are given polynomial matrices

axbyc

Solvability

Equation $AX + BY = C$ has a solution if and only if

$\gcd(A, B)$ is a left divisor of *C*

$$\mathrm{gcld}(A,B)\,|_{\mathrm{L}}$$

General solution

Let *X'*, *Y'* is a particular solution and let $n = \mathrm{rank}[A, B]$.

Then the general solution is

$$X = X' - B_1 T$$
$$Y = Y' + A_1 T$$

where $B_1 \in \mathrm{R}_{p,p+q-n}[s], A_1 \in \mathrm{R}_{q,p+q-n}[s]$ are right coprime matrices satisfying $AB_1 = BA_1$ and $T \in \mathrm{R}_{p+q-n,m}[s]$ is an arbitrary polynomial matrix.

xaybc

Solvability

Equation $XA + YB = C$ has a solution if and only if

$\mathrm{gcrd}(A, B)$ is a right divisor of *C*

$\mathrm{gcld}(A, B)|_{\mathrm{R}}$

General solution

Let *X'*, *Y'* is a particular solution and let $n = \operatorname{rank}\begin{bmatrix} A \\ B \end{bmatrix}$.

Then the general solution is

$$\begin{aligned} X &= X' - TB_2 \\ Y &= Y' + TA_2 \end{aligned}$$

where $B_2 \in \mathrm{R}_{p+q-n,p}[s], A_2 \in \mathrm{R}_{p+q-n,q}[s]$ are left coprime matrices satisfying $B_2 A = A_2 B$ and $T \in \mathrm{R}_{l,p+q-n}[s]$ is an arbitrary polynomial matrix.

Algorithms

- Block Sylvester – OK but no good guess about deg
- Elementary operations: to be shown for axbyc:
  - Apply elementary column operation to reduce

$$\begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix} \rightarrow \begin{bmatrix} G_1 & 0 \\ P_1 & V_1 \\ Q_1 & W_1 \end{bmatrix}$$

  - Compute $C_1 : \; C = G_1 C_1$
  - Then

$$X = P_1 C_1 + V_1 T$$
$$Y = Q_1 C_1 + W_1 T$$

Bilateral equation

The most general linear matrix equation to be used in the course is

$$A(s)X(s)+B(s)Q(s)=C(s)$$

axybc

where $A \in \mathbb{R}_{lp}[s], B \in \mathbb{R}_{qm}[s], C \in \mathbb{R}_{lm}[s]$

- This equation is two-sided!
- It is much more difficult to solve.

Solvability (not constructive)

The equation is solvable iff the matrices

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

are equivalent.

Computation: Make one-sided using Kronecker product