# Polynomial Methods for Control Analysis and Design

[ PolyX ]

**1/ Polynomials and polynomial matrices**

## Overview

[ PolyX ]

## Overview

### Ch. 2. Polynomials and Polynomial Toolbox

Preview
Polynomial
Polynomial Toolbox 2
Fields and rings
Ring of polynomials
Polynomial fractions
Polynomial equation
Polynomial matrices
Polynomial matrix fractions
Polynomial matrix equations



## Preview: polynomials in control

Transfer function and matrix
Feedback loop
Simple analysis
Simple synthesis
Polynomial equation

## Transfer function and transfer matrix

[ PolyX ]

SISO systems

Often described by rational transfer functions looking like

$$\frac{b(s)}{a(s)}$$

But this is nothing else than a polynomial fraction.

MIMO systems

Often described by rational transfer matrix looking like

$$\begin{bmatrix} \frac{b_{11}(s)}{a_{11}(s)} & \cdots \\ \cdots & \ddots \end{bmatrix}$$

But this is nothing else than a matrix of polynomial fractions.

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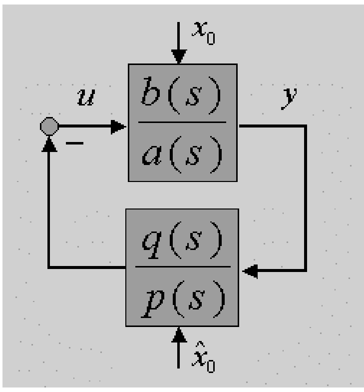
## Feedback loop

[ PolyX ]

If there are no hidden modes in the plant and controller descriptions, then

$$a(s)p(s)+b(s)q(s)$$

is the characteristic polynomial of the closed loop

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## Analysis

Task A: Analysis

Given plant $a(s), b(s)$ and controller $p(s), q(s)$ ,

compute $a(s)p(s) + b(s)q(s)$ and analyze it.

May check

- stability
- position of poles
- robustness
- ....



## Synthesis

Task B: Synthesis

Desired c-l characteristic polynomial

Given plant $a(s), b(s)$ and a polynomial $c(s)$ ,

compute a controller $p(s), q(s)$ so that

$$a(s)p(s) + b(s)q(s) = c(s)$$

The resulting controller guarantees that the closed-loop characteristic polynomial is $c(s)$ and hence has the desired properties!

## Polynomial equation

Basic problem:

Given $a(s), b(s)$ and $c(s)$ , how one gets $p(s), q(s)$

such that $a(s)p(s) + b(s)q(s) = c(s)$ ???

It must be solved as an equation in polynomials !

This is a LINEAR POLYNOMIAL EQUATION



## Polynomial

Various representations

# Polynomial by coefficients

[ PolyX ]

Polynomial represented by coefficients

coefficients

degree

$$p(s) = p_0 + p_1 s + \cdots + p_n s^n$$

$\deg p(s) = n$
$\Leftrightarrow p_n \neq 0;$
$\deg 0 = -\infty$

constant coefficient

variable, indeterminate

leading coefficient

the representation used in the Polynomial Toolbox



# Polynomial by values

[ PolyX ]

Polynomial represented by values

$n + 1$ distinct complex points

$$p(s_0),\, p(s_1),\, p(s_2), \ldots,\, p(s_n) \in C$$

$n + 1$ complex values

## Polynomial by roots

Polynomial represented by its roots

- if complex numbers allowed

$n$ complex roots

$$p(s) = c(s - s_1)\cdots(s - s_n)$$

- if complex numbers are not allowed, then represent each complex conjugate pair by the second degree polynomial

$$s^2 - 2\alpha s + \alpha^2 + \beta^2 = (s - (\alpha + i\beta))(s - (\alpha - i\beta))$$



## Polynomial Toolbox 2

Key features
Local installation
Initialization
poldesk
Manual

## Polynomial Toolbox

### Polynomial Toolbox 2.0

[ PolyX ]

- Based on Matlab v. 5 or later
- Object oriented:
  polynomials and polynomial matrices
  are defined as objects
- More than 200 routines
- Stand-alone but easily cooperates with
  Simulink, Control Systems Toolbox,
  Symbolic Math Toolbox



## Key features

### Key Features

- Simple input, manipulation and display of polynomials and polynomial matrices based on a new polynomial matrix object
- Overloaded operations and functions, solvers for numerous linear and quadratic matrix polynomial equations
- Polynomial matrices with complex coefficients for applications in signal processing
- New generation of numerical algorithms: easy, fast, reliable
- Polynomial Matrix Editor, 2-D and 3-D color plots

## Key features - 2

- Continuous-time and discrete-time system and signal models based on polynomial matrix fractions
- Classical and robustness analysis for LTI systems and filters
- Classical and optimal design tools: pole placement, all stabilizing controllers, dead-beat, H2 and LQG
- H-infinity optimization in a generality not found elsewhere
- Robust control with parametric uncertainties: single parameter, interval and poly-topic



## Key features - 3

- Conversion to and from LTI objects of the Control System Toolbox and polynomial objects defined in the Symbolic Math Toolbox
- Simulink block set for LTI systems described by polynomial matrix fractions

## Local installation

- ?



## Initialization

Initialization command

```
» pinit

Polynomial Toolbox initialized. To get started, type one of
these: helpwin or poldesk. For product information, visit
www.polyx.com or www.polyx.cz.
```

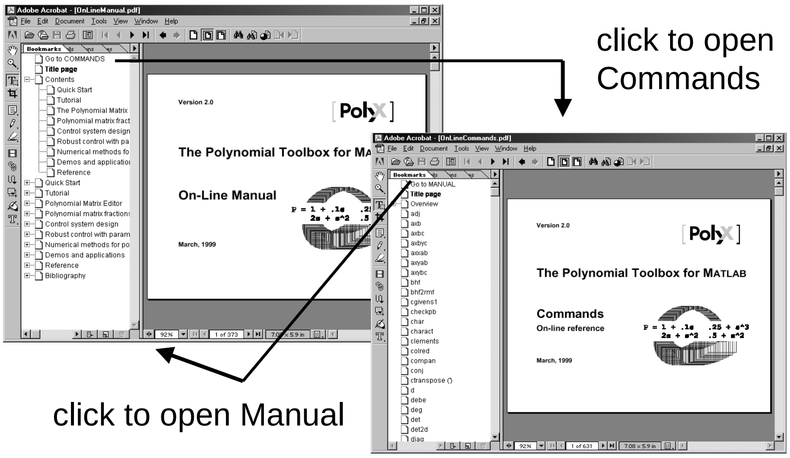
On line help: in pdf

PolyX Webs

On line help: in Matlab help window

# poldesk

» poldesk

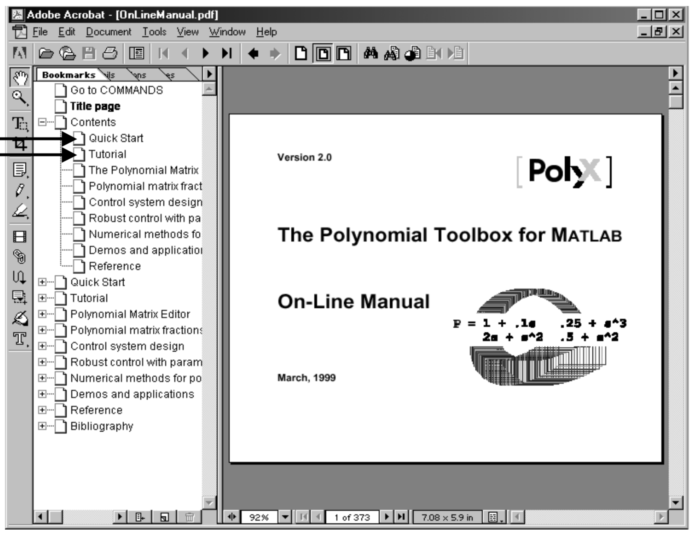



click to open Commands

click to open Manual



# Manual



# Rings and fields

Rings
Groups
Commutative rings
Units and fields
Divisors and multiples
Primness and coprimness

## Rings

Ring

Set G equipped with two operations

- addition
- multiplication

which is

- a commutative group with respect to addition
- a semigroup with respect to multiplication

such that

- multiplication distributes over addition

## Groups

Group with respect to addition

- the addition is associative
- the addition is commutative
- a neutral element 0 exists
- the additive inverse exists

Semigroup with respect to multiplication

- multiplication is associative
- multiplicative neutral element 1 exists



## Commutative rings

Commutative ring

Ring such that multiplication is commutative

Examples of commutative rings

- the set of integers
- the set of all polynomials

## Units and fields

If an element $\boldsymbol{u}$ which belongs to a ring has a multiplicative inverse then $\boldsymbol{u}$ is called a unit

Examples:
- The set of integers has the units +1 and –1
- Units of the set of polynomials are nonzero real numbers

**Field**
Commutative ring in which the zero and unit element are different and each nonzero element is a unit

Examples:
- The rationals, the reals, the complex numbers, the rational functions



## Divisors and multiples

Notions for elements of a commutative ring

- $a$ is a divisor of $b$ if there exists a $c$ such that $b = ac$
- $a$ is a multiple of $b$ if there exists a $c$ such that $a = bc$
- If $g$ divides both $a$ and $b$ then it is a common divisor of $a$ and $b$
- If $g$ is a multiple of every common divisor of $a$ and $b$ then it is a greatest common divisor
- If $l$ is a multiple of both $a$ and $b$ then it is called a common multiple of $a$ and $b$
- If $l$ is a divisor of every common multiple of $a$ and $b$ then it is a least common multiple of $a$ and $b$

## Coprimness and primness

- $a$ and $b$ are coprime if their only common divisors are units of the ring
- $a$ is prime if it is divisible only by units and elements of the form $au$, with $u$ a unit



## Ring of polynomials

Features
Euclidean division
GCD and LCM
Roots
Stability

## Features

Ring of polynomials in $s$: $\mathbb{R}[s]$, $\mathbb{C}[s]$

( $\mathbb{R}(s)$, $\mathbb{C}(s)$ are sequences or fractions)

- units = polynomials of degree 0 (isomorphic with nonzero constants)
- other polynomials are not units so it is only a ring not a field!
- it has no zero divisors (no $a,b\neq 0: ab=0$ ) so it is an integral domain
- prime elements = irreducible polynomials in $\mathbb{R}[s]$ of the form $p_0+p_1s$ or $p_0+p_1s+p_2s^2,\ p_1^2-4p_0p_2<0$



## Euclidean division


Euclid (Alexandria, ~ 300 BC)

- basic tool for studying divisibility

Euclidean ring

Given polynomials $a,b$ with $b\neq 0$
polynomials $q,r$ exist
such that

$$\boxed{a = bq + r,\ \deg r < \deg b}$$

quotient remainder

## GCD and LCM

Formulas for GCD and LCM

For any polynomials $a, b \in \mathbb{C}[s]$ we can always find their gcd $g$ and lcm $l$ along with two pairs of coprime polynomials $p, q$ and $v, w$ such that

$$ap + bq = g$$
$$av + bw = 0$$

and

$$l = av = -bw$$

PID
Principal Ideal Domain

(not true for polynomials in two variables)



## Roots

Definition:

Polynomial $p(s)$ has a root $s_i \in \mathrm{C}$ if $p(s_i) = 0$

Facts:

- Fundamental Theorem of Algebra
  A polynomial of degree $n$
  $$p(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n, \quad p_n \neq 0$$
  has exactly $n$ roots (when counted with multiplicities)
- Complex conjugate pairs
  If all the coefficients $p_i$ are real, then complex roots come in complex conjugate pairs $(\alpha + i\beta, \alpha - i\beta)$

## Stability

Location of roots

- plays an important role in technical applications of polynomials
- reveals stability and other dynamical properties of underlying systems

A polynomial is

- Hurwitz stable iff all its roots are in open left half plane
- Schur stable iff all its roots are inside of unit disc
- Stable in $z^{-1}$ or $d$ - " - outside - " –
- *D*-stable iff all its roots are in *D*



## Notation and formulas

| | |
|---|---|
| $a \mid b$ | stands for $a$ divides $b$ |
| $\gcd(a,b) = (a,b)$ | two options denoting gcd |

$$\deg a(s)b(s) = \deg a(s) + \deg b(s)$$

## Polynomial fractions

Field of fractions
Properness
Causality
Poles and zeros
Stability

## Field of fractions

Consider the ring of polynomials $\mathbb{R}[s]$ and fractions of the form

$$a = \frac{a_2}{a_1}, \; a_1, a_2 \in \mathbb{R}[s], a_1 \neq 0$$

$$b = \frac{b_2}{b_1}, \; b_1, b_2 \in \mathbb{R}[s], b_1 \neq 0$$

With equality $a = b$ defined by $a_2 b_1 = b_2 a_1$ and addition and multiplication by

$$a + b = \frac{a_2 b_1 + a_1 b_2}{a_1 b_1}$$

$$ab = \frac{a_2 b_2}{a_1 b_1}$$

the set of equivalence classes of these fractions is a field. Each class can be characterized by a fraction with coprime numerator and denominator

## Properness

Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

is

- proper if $\deg p(s) \geq \deg q(s)$
- strictly proper if $\deg p(s) > \deg q(s)$
- improper if $\deg p(s) < \deg q(s)$
- biproper if $\deg p(s) = \deg q(s)$



## Causality

Polynomial fraction

$$h(d) = \frac{b(d)}{a(d)}$$

is

- causal if $a(0) \neq 0$
- strictly causal if $a(0) \neq 0, b(0) = 0$
- improper if $a(0) = 0, b(0) \neq 0$
- biproper if $a(0) \neq 0, b(0) \neq 0$

## Poles and zeros

Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

- Poles: Roots of $p$
- Zeros: Roots of $q$



## Poles and zeros - 2

Definition:

- Poles/zeros at Infinity
  Polynomial fraction

$$h(s) = \frac{q(s)}{p(s)}$$

has a pole/zero at infinity iff the polynomial fraction

$$h(1/s)$$

has a pole/zero at $0$.

## Poles and zeros - 3

Facts: about $h(s)=\dfrac{q(s)}{p(s)}$

- If $\deg p(s) > \deg q(s)$ then it has at infinity a zero with multiplicity $k = \deg\ p(s) - \deg q(s)$
- If $\deg p(s) < \deg q(s)$ then it has at infinity a zero with multiplicity $l = \deg q(s) - \deg\ p(s)$
- When taking poles/zeros at infinity into account, then

  Number of poles = Number of zeros



## Stability

Polynomial fraction

$$h(s)=\frac{q(s)}{p(s)}$$

- is stable iff its denominator *p* is a stable polynomial

## Polynomial equations

Polynomial equation
Solvability
General solution
Minimum degree solution
Coincidence
Some algorithms

## Preview

Equations in a field look like this

$\boldsymbol{ax=b}$

and are solved by like this

$\boldsymbol{x=a\backslash b}$

This is hardly solvable in a field !

(unless $\boldsymbol{a}$ is a unit)

More useful is

$\boldsymbol{ax+by=c}$

## Polynomial equations

Linear polynomial equations - also called Diophantine

(in integers by Diophantus from Alexandria, ~300 AD)

Given polynomials

no picture found

$$a(s)x(s)+b(s)y(s)=c(s)$$

Polynomials to be computed



## Solvability

Solvability condition

The equation

$$a(s)x(s)+b(s)y(s)=c(s)$$

is solvable if and only if

$$\gcd(a,b)\,|\,c$$

(the greatest common divisor of $a$ and $b$ divides $c$)

## Proof

Only if:

Let $ax'+by'=c$

and write $(a,b)=g,\ a=g\bar{a},\ b=g\bar{b}$

Then $g(\bar{a}x'+\bar{b}y')=c$ and hence $g\,|\,c$ ☺

If:

Let $(a,b)\,|\,c$

and denote $(a,b)=g,\ c=g\bar{c}$

Then there always exist $p,q$ such that $ap+bq=g$

Multiplying by $\bar{c}$ we get $a(p\bar{c})+b(q\bar{c})=c$

and hence he have constructed a solution

$$x=p\bar{c},\ y=q\bar{c}$$ ☺



## General solution

Denoting $\bar{a}=\dfrac{a}{\gcd(a,b)},\quad \bar{b}=\dfrac{b}{\gcd(a,b)}$

the general solution reads

Note that "-" can be "everywhere"

$$x(s)=x'(s)-\bar{b}(s)t(s)$$
$$y(s)=y'(s)+\bar{a}(s)t(s)$$

solutions are infinitely many!

particular solution

arbitrary polynomial parameter

## Proof

Proof:
By assumption $ax'+by'=c$
and hence $a(x-x')+b(y-y')=0$ .
Now polynomials $\bar{a},\bar{b}$ defined before are coprime
and satisfy $\bar{a}b=a\bar{b}$ .
As a result $\bar{b} \mid x-x'$ and $\bar{a} \mid y-y'$ , that is,

$$x-x'=-\bar{b}t$$
$$y-y'=\bar{a}t$$

for some polynomial $t$ . To obtain any solution, we let
$t$ range over $\mathbb{R}[s]$ and the claim follows. ☺



## Minimum degree solutions

Take the general solution $x=x'-\bar{b}t$
and use the division $y=y'+\bar{a}t$
algorithm to reduce
$x'$ modulo $\bar{b}$ : $x'=\bar{b}q+r, \quad \deg r<\deg \bar{b}$
Then $x=r-\bar{b}(t-q)$
and the minimum degree solution $x,y$ wrt $x$ becomes

$$\boxed{\begin{aligned} x&=r \\ y&=y'+\bar{a}q \end{aligned}} \qquad \boxed{\deg x<\deg \bar{b}}$$

either
is unique

There is a(other) minimum degree solution
$x,y$ wrt $y$ characterized by $\boxed{\deg y<\deg \bar{a}}$

but may
not be
identical

## Coincidence condition

[ PolyX ]

An important condition

$$(a,b)=1:\ ax+by=c\ \mapsto\ \boxed{\frac{x}{b}}+\boxed{\frac{y}{a}}=\frac{c}{ab}$$

s.p. if sol. is min.deg.wrt $x$

s.p. if sol. is min.deg.wrt $y$

s.p. if $\deg c<\deg a+\deg b$

If RHS s.p. then either both LHE are s.p. or none!

If RHS is not s.p., then only one of LHE can be s.p. at a time!



## Coincidence

[ PolyX ]

One minimum degree solution

If $\deg c<\deg a+\deg b$
then both minimum degree solutions coincide
and hence there exists only one min. deg. sol.
(which has min. degree of both x and y!)

Otherwise, if $\deg c\geq\deg a+\deg b$
then there are really two different minimum degree
solutions.

## Some algorithms

How can you solve polynomial equations?

Use the Polynomial Toolbox !

Only if unplugged, try (for simple examples) your pencil (and back side of an envelope mentioned by Mrs. A. Einstein) and one of these routines:

- Solution via elementary operations (reductions)
- Solution using Sylvester matrix

to be described.

Other methods not explained here include

- State-space solution or
- Interpolation

.



## Polynomial reductions

Elementary operations on a polynomial matrix

Row operations: 3 basic

- multiplying a row by a nonzero constant

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{multiply first row by 3}} \begin{bmatrix} 3 & 3s \\ 2 & s^2 \end{bmatrix}$$

- interchanging two rows

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow{\text{interchange the first and second row}} \begin{bmatrix} 2 & s^2 \\ 1 & s \end{bmatrix}$$

## Polynomial reductions - 2

- adding a polynomial multiple of one row to another

$$\begin{bmatrix} 1 & s \\ 2 & s^2 \end{bmatrix} \xrightarrow[\text{and add the result to the first row}]{\text{multiply the second row by } s} \begin{bmatrix} 1+2s & s+s^3 \\ 2 & s^2 \end{bmatrix}$$

Column operations are dual

(Elementary operations preserve determinant
= multiplication by a unimodular matrix)



## Solution via polynomial reductions

Solution of $a(s)x(s)+b(s)y(s)=c(s)$ via reductions

- Step 1 Form a composite matrix

$$\begin{bmatrix} a(s) & 1 & 0 \\ b(s) & 0 & 1 \end{bmatrix}$$

- Step 2 Reduce it via elementary row ops.

$$\begin{bmatrix} g(s) & p(s) & v(s) \\ 0 & q(s) & w(s) \end{bmatrix}$$

## Solution via polynomial reductions - 2

Then $$\begin{aligned} p(s)a(s)+q(s)b(s) &= g(s) \quad \text{gcd}(a,b) \\ v(s)a(s)+w(s)b(s) &= 0 \end{aligned}$$

- Step 3 Extract $g(s)$ from $c(s)$ to get

$$c(s)=\bar{c}(s)g(s)$$

If not possible ➡ **stop!**

- Step 4 Take

$$\begin{aligned} x(s)&=\bar{c}(s)p(s) \\ y(s)&=\bar{c}(s)q(s) \end{aligned}$$



## Solution via polynomial reductions - 3

- Moreover, all solutions are expressed as

$$\begin{aligned} x(s)&=\bar{c}(s)p(s)+v(s)t(s) \\ y(s)&=\bar{c}(s)q(s)+w(s)t(s) \end{aligned}$$

free
polynomial
parameter

## Solution via Sylvester matrix - 1

Solution of $a(s)x(s)+b(s)y(s)=c(s)$ via reductions

The method will be explained for a simple case with given

$$\begin{aligned} a(s) &= a_0 + a_1 s + a_2 s^2 \\ b(s) &= b_0 + b_1 s + b_2 s^2 \\ c(s) &= c_0 + c_1 s + c_2 s^2 \end{aligned}$$

and expected

$$\begin{aligned} x(s) &= x_0 + x_1 s \\ y(s) &= y_0 + y_1 s \end{aligned}$$

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## Solution via Sylvester matrix - 2

- Step 1 Expand and equate coefficients at like powers to get a linear equation with constant matrices

$$\underbrace{\begin{bmatrix} x_0 & y_0 & x_1 & y_1 \end{bmatrix}}_{\text{unknowns}} \begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ b_0 & b_1 & b_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 & b_2 \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & c_2 & 0 \end{bmatrix}$$

Sylvester, resultant matrix

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## Solution via Sylvester matrix - 3

[ PolyX ]

- Step 2 Solve the Sylvester equation

$$\Rightarrow \quad x_0, x_1, y_0, x_0$$

- Step 3 Construct the polynomials

$$\begin{aligned} x(s) &= x_0 + x_1 s \\ y(s) &= y_0 + y_1 s \end{aligned}$$



## Bezout equation

[ PolyX ]

Bezout equation

- looks like this $a(s)x(s)+b(s)y(s)=1$
- is a particular case of Diophantine equation
- is solvable iff $a$ and $b$ are coprime
- has unique minimum degree solution
  as coincidence is guaranteed

## Open problems

### The Diophantine equations

- have been intensively studied: over decades in the polynomial case, or even over millennia in the integer case
- so they are well understood yet,
- surprisingly, some problems are still open

Open problems are e.g.

- What is real degree of the min. deg. sol.? The condition $\deg x < \deg \bar{b}$ generically reads $\deg x = \deg \bar{b} - 1$ but may be $\deg x < \deg \bar{b} - 1$ When and why?
- When the solution fails to be coprime?
- When the solution satisfies additional conditions (e.g. $x$ stable) and how to find it?



## Polynomial matrices

Polynomial matrices
Divisibility
Coprimness and primness
GCD and LCM
Division with reminder
Zeros

## Polynomial matrices

Polynomial matrices are matrices over the ring of polynomials.
There are two ways how to look at a polynomial matrix $P(s)$

1) P is a matrix whose entries are polynomials

$$P(s) = \begin{bmatrix} 1+s & s^2 \\ 3+s^3 & 4 \end{bmatrix}$$

2) P is a polynomial whose coefficients are matrices

$$P(s) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s^3$$



## Algebraic structure

- Square polynomial matrices $m \times m,\ m > 1$ form a noncommutative ring
- A square matrix $A(s)$ is a unit in the ring iff $\det A(s)$ is a unit in the ring of scalar polynomials, that is, iff $\det A(s)$ is a nonzero constant. Such matrix is called unimodular.
- A square matrix $A(s)$ is a zero divisor iff $\det A(s) = 0$

# Divisibility

Consider polynomial matrices *A, B* and *C.* If *A=BC,* then

- *B* is a left divisor of *A*
- *A* is a right multiple of *B*
- *C* is a right divisor of *A*
- *A* is a left multiple of *C*
- If *B* is square unimodular, then *A* and *C* are left equivalent
- If *C* is square unimodular, then *A* and *B* are right equivalent
- If *A=BCD* with both *B* and *D* square unimodular, then *A* and *C* are equivalent.

Consider now polynomial matrices *A* and *B* with the same number of rows (columns):

- If $G_1$ is a left (right) divisor of both *A* and *B,* then it is their common left (right) divisor
- If, furthermore, $G_1$ is a right (left) multiple of every common left (right) divisor of *A* and *B,* then it is their greatest common left (right) divisor.



# Coprimness

- Matrices *A,B* with the same number of rows (columns) are left (right) coprime if their only common left (right) divisors are unimodular matrices.
- A matrix *A* is left (right) prime if their only left divisors are unimodular matrices.
- Matrices *A,B* are left coprime if the composite matrix [*A,B*] is left prime and vice versa.
- Matrices *A,B* are right coprime if the composite matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is right prime and vice versa.
- Such a way, coprimness can easily be generalized for 3 and more matrices.

## GCD and LCM – matrix case

Formulas for matrix GCD and LCM

- For any two polynomial matrices *A,B* with the same number of rows, we can always find their gcld $G_1$ and lcrm $L_1$ along with two pairs of right coprime matrices $P_1$, $Q_1$ and $V_1$, $W_1$ such that

$$\begin{aligned} AP_1 + BQ_1 &= G_1 \\ AV_1 + BW_1 &= 0 \end{aligned}$$ and $$L_1 = AV_1 = -BW_1$$

- For any two polynomial matrices *A,B* with the same number of columns, we can always find their gcrd $G_2$ and lclm $L_2$ along with two pairs of right coprime matrices $P_2$, $Q_2$ and $V_2$, $W_2$ such that

$$\begin{aligned} P_2A + Q_2B &= G_2 \\ V_2A + W_2B &= 0 \end{aligned}$$ and $$L_2 = V_2A = -W_2B$$



## Division with a remainder for matrices

Left division:

- Consider two matrices *N,D* with the same number of rows, *D* square and full rank. Then there exist unique polynomial matrices *Q,R* such that

$N = DQ + R$ and $D^{-1}R$ is a strictly proper rational matrix

- This division corresponds to splitting of a rational matrix into a polynomial and strictly proper part

$$D^{-1}N = Q + N^{-1}R$$

rational

polynomial part

strictly proper part

- Right division with a reminder is defined dually.

## Zeros

Zero

- of a polynomial matrix $P$ is defined as a complex number $s_i$ such that

$$\operatorname{rank} P(s_i) < \operatorname{rank} P(s)$$

- If $P$ is square full rank, then its zeros equal roots of

$$\det P(s)$$



## Polynomial matrix fractions

Left and right fractions
Conversion rational-pmf
Conversion lmf-rmf

## Polynomial matrix fractions

How can we generalize 'polynomial fraction' for matrices?

- A rational matrix $F$ can be described by means of two polynomial matrices as

left fraction

$$F(s) = D(s)^{-1} N(s)$$

polynomial matrix numerator

left polynomial matrix denominator (square, invertible)

fraction is coprime if $D,N$ are so

- Alternatively, it can be described by means of other two polynomial matrices as

right fraction

$$F(s) = N(s) D(s)^{-1}$$

right polynomial matrix denominator (square, invertible)

polynomial matrix numerator



## Conversions: rational and PMF

Rational to left PMF:

1) Take least common denominator in each row of $F$
2) Write $F(s) = diag\{1/d_i(s)\} N(s) = \underbrace{diag\{d_i(s)\}}_{D(s)}^{-1} N(s)$
3) Make it coprime if required.

Left PMF to rational:

$$D^{-1}(s)N(s) = \frac{1}{\det D(s)} \overbrace{(\operatorname{adj} D(s))\, N(s)}^{F_N(s)} = \left[ \frac{f_{N,ij}}{\det D} \right]$$

and make each entry coprime if required.

For Right PMF similarly.

# Conversions: left-right

## Left to right PMF

Given a left PMF $D^{-1}(s)N(s)$ , compute right PMF as follows:

- Make a composite matrix $[D(s), N(s)]$
- Solve the equation (find a right null space, in fact)

$$[D(s), N(s)]\begin{bmatrix} Y(s) \\ X(s) \end{bmatrix} = 0$$

- Then the desired right fraction reads $Y(s)X^{-1}(s)$

## Right to left PMF

Given a right PMF $N(s)D^{-1}(s)s$

- Make a composite matrix $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ and solve $[X(s), Y(s)]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$
- Then the desired right fraction reads $X^{-1}(s)Y(s)$



# Polynomial matrix equations

axbyc
xaybc
axybc

## Matrix equations in polynomials

[ PolyX ]

A natural generalization of

$$a(s)x(s)+b(s)q(s)=c(s)$$

is either

$$A(s)X(s)+B(s)Y(s)=C(s)$$ axbyc

where $A\in \mathbb{R}_{lp}[s], B\in \mathbb{R}_{lq}[s], C\in \mathbb{R}_{lm}[s]$
are given polynomial matrices or

$$X(s)A(s)+Y(s)B(s)=C(s)$$ xaybc

where $A\in \mathbb{R}_{pm}[s], B\in \mathbb{R}_{qm}[s], C\in \mathbb{R}_{lm}[s]$
are given polynomial matrices



## axbyc

[ PolyX ]

### Solvability

Equation $AX+BY=C$ has a solution if and only if
$\mathrm{gcld}(A,B)$ is a left divisor of *C*

$$\mathrm{gcld}(A,B)|_{\mathrm{L}}$$

### General solution

Let *X'*, *Y'* is a particular solution and let $n=\mathrm{rank}[A,B]$ .
Then the general solution is

$$\begin{aligned} X &= X'-B_1T \\ Y &= Y'+A_1T \end{aligned}$$

where $B_1\in \mathbb{R}_{p,p+q-n}[s], A_1\in \mathbb{R}_{q,p+q-n}[s]$ are right coprime matrices satisfying $AB_1=BA_1$ and $T\in \mathbb{R}_{p+q-n,m}[s]$ is an arbitrary polynomial matrix.

# xaybc

## Solvability

Equation $XA + YB = C$ has a solution if and only if $\mathrm{gcrd}(A,B)$ is a right divisor of $C$

$\mathrm{gcld}(A,B)|_{\mathrm{R}}$

## General solution

Let $X', Y'$ is a particular solution and let $n = \mathrm{rank}\begin{bmatrix} A \\ B \end{bmatrix}$.
Then the general solution is

$$\begin{aligned} X &= X' - TB_2 \\ Y &= Y' + TA_2 \end{aligned}$$

where $B_2 \in \mathsf{R}_{p+q-n,p}[s], A_2 \in \mathsf{R}_{p+q-n,q}[s]$ are left coprime matrices satisfying $B_2 A = A_2 B$ and $T \in \mathsf{R}_{l,p+q-n}[s]$ is an arbitrary polynomial matrix.



# Algorithms

- Block Sylvester – OK but no good guess about deg
- Elementary operations: to be shown for axbyc:
  - Apply elementary column operation to reduce

$$\begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix} \rightarrow \begin{bmatrix} G_1 & 0 \\ P_1 & V_1 \\ Q_1 & W_1 \end{bmatrix}$$

  - Compute $C_1 : \; C = G_1 C_1$
  - Then

$$\begin{aligned} X &= P_1 C_1 + V_1 T \\ Y &= Q_1 C_1 + W_1 T \end{aligned}$$

# Bilateral equation

[PolyX]

The most general linear matrix equation to be used in the course is

$$A(s)X(s)+B(s)Q(s)=C(s)$$ axybc

where $A\in \mathsf{R}_{lp}[s], B\in \mathsf{R}_{qm}[s], C\in \mathsf{R}_{lm}[s]$

- This equation is two-sided!
- It is much more difficult to solve.

Solvability (not constructive)

The equation is solvable iff the matrices

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

are equivalent.

Computation: Make one-sided using Kronecker product