# Polynomial Methods for Control Analysis and Design

[ PolyX ]

5/ Continuous-time
and
MIMO systems

# Overview

# Overview

## Ch. 5. Continuous-time and MIMO systems

C-t systems and polynomials

Feedback systems

Feedback design

Tracking problems

MIMO systems

Linear quadratic regulator

Linear Gaussian filter

Spectral factorization

LQG

$H_2$ optimization

# Continuous-time systems and polynomials

???

# SS and IO representation

[PolyX]

## Computation of the IO representation

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

$$y = Cx + Du$$

Using Laplace transform $\mathcal{L}(\dot{x}) = sx(s) - x(0) = sx(s) - x_0$

$$y(s) = \left(C(sI - A)^{-1}B + D\right)u(s) + C(sI - A)^{-1}x_0$$

$$y(s) = \frac{q(s)}{p(s)}u(s) + \frac{r_{x_0}(s)}{p(s)}$$

$r_{x_0}$ is not used in design

$$p(s) = \det(sI - A)$$

# Hidden modes

## Hidden modes:

Let $p(\frac{d}{dt})y = q(\frac{d}{dt})u$ is IO model of $\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$

that is $\frac{q(s)}{p(s)} = \left(C(sI - A)^{-1}B + D\right), \; \frac{r_{x_0}(s)}{p(s)} = C(sI - A)^{-1}x_0$

We say that it has no hidden modes iff

$$p(s) = \det(sI - A)$$

## Equivalent expressions:

- Nothing has been cancelled during computation
- “What you see is what you have”

# Convention

It is always assumed that there is
no common factor present (at the same time)
in all the polynomials $p, q, r_{x_0}$

Remember that $r_{x_0}$ is a “set” !

Then no hidden modes requirement means

- the system is observable
- the system is controllable iff $p$ and $q$ are coprime

# Plant

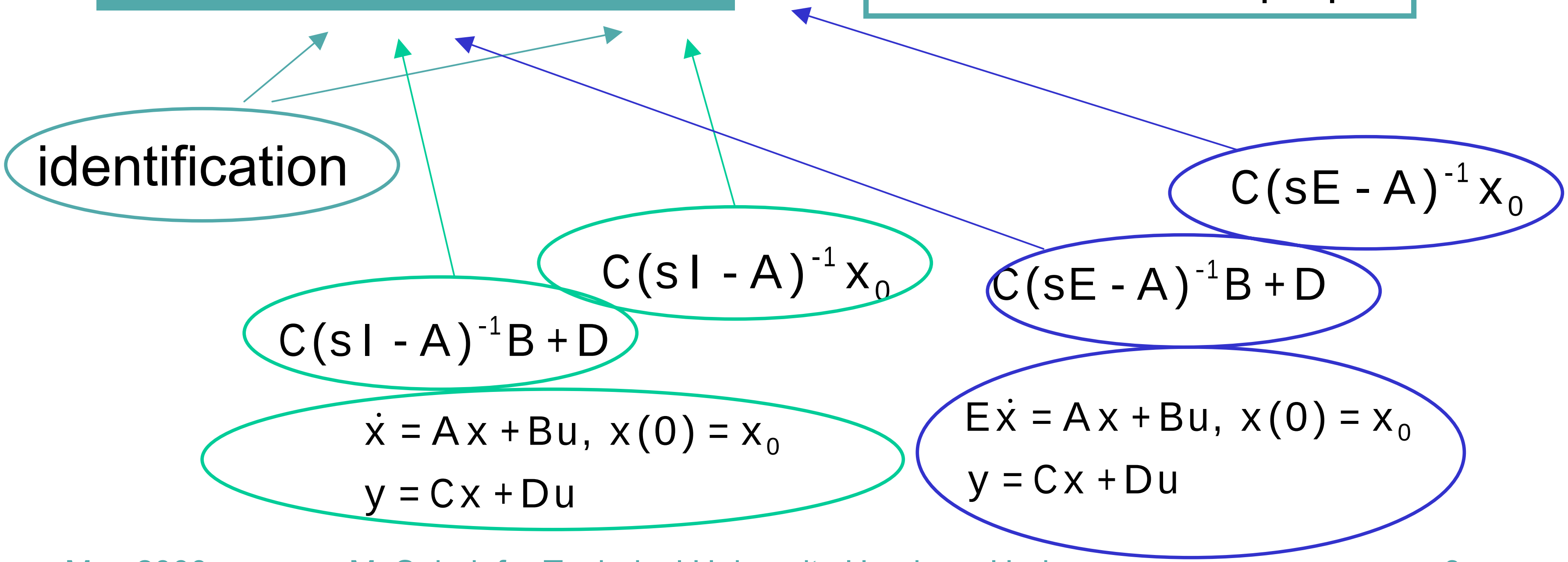
## Plant

The given system to be controlled

- No hidden modes
- Don't care about $c_{x_0}$

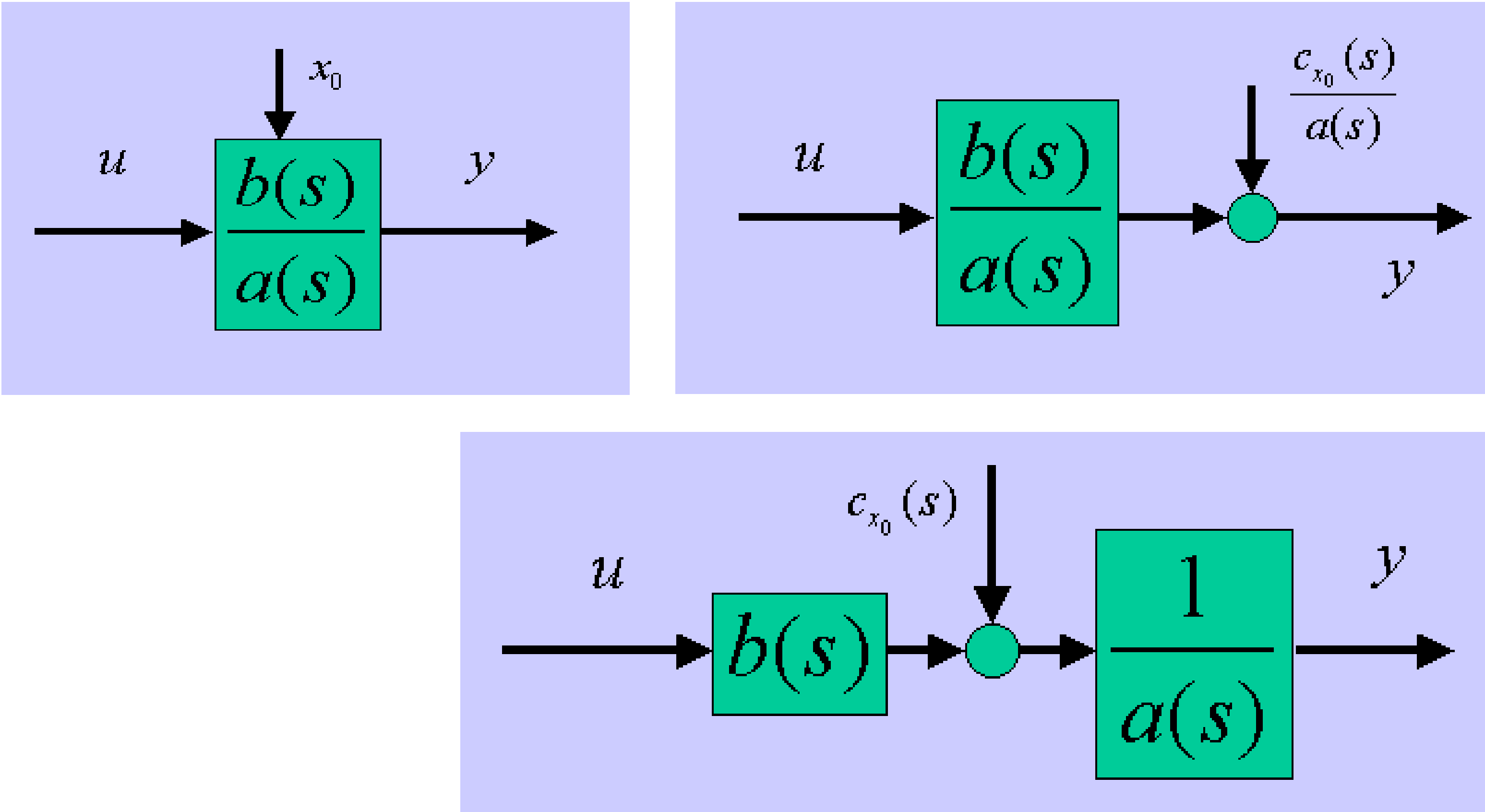
$$y(s) = \frac{b(s)}{a(s)} u(s) + \frac{c_{x_0}(s)}{a(s)}$$

**New assumption:**
plant is proper, i.e.
both fractions are proper



# Plant

## Some equivalent pictures



# Controller

## Controller

The system to be found

- To meet design specs.
- Realized without hidden modes
- $r_{x_0}$ can be any

$$u(s) = \frac{q(s)}{p(s)} y(s) + \frac{r_{\hat{x}_0}(s)}{p(s)}$$

New requirement:
controller is proper =
both fractions are proper

directly

$\hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$

$\hat{C}(sI - \hat{A})^{-1}\hat{x}_0$

$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y, \quad \hat{x}(0) = \hat{x}_0$

$-u = \hat{C}\hat{x} + \hat{D}y$

$\hat{C}(sE - \hat{A})^{-1}\hat{x}_0$

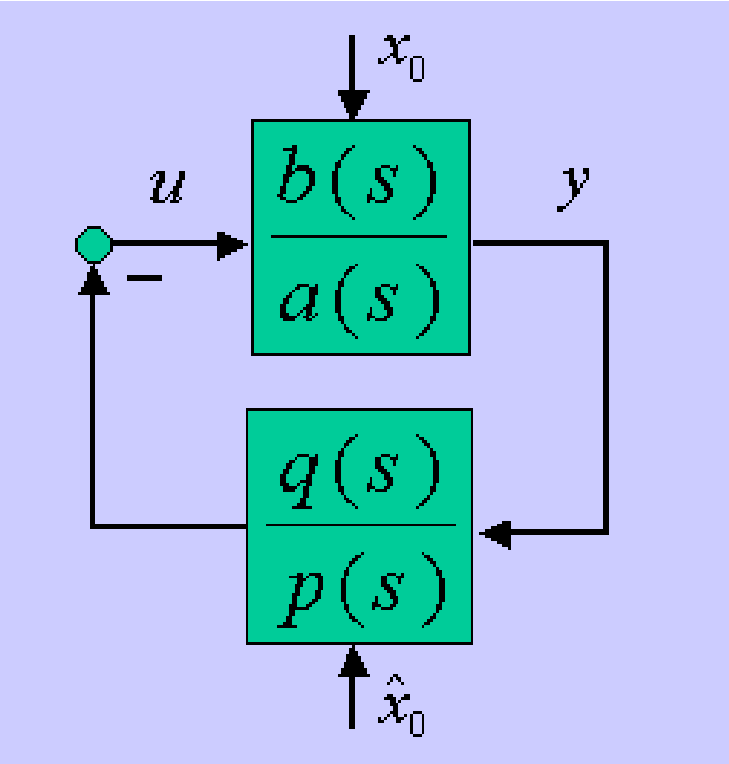
$\hat{C}(sE - \hat{A})^{-1}B + D$

$E\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}y, \quad \hat{x}(0) = \hat{x}_0$

$-u = \hat{C}\hat{x} + \hat{D}y$

# Feedback systems

Consider a simple feedback system




Plant

$$y(s) = \frac{b(s)}{a(s)} u(s) + \frac{c_{x_0}(s)}{a(s)}$$

Controller

$$u(s) = -\frac{q(s)}{p(s)} y(s) + \frac{r_{\hat{x}_0}(s)}{p(s)}$$

Important assumption: No hidden modes !

# Feedback systems

$$y = \frac{b}{a}u + \frac{c_{x_0}}{a}$$

$$y = -\frac{bq}{ap}y + \frac{br_{\hat{x}_0}}{ap} + \frac{pc_{x_0}}{ap}$$

$$(ap + bq)y = br_{\hat{x}_0} + pc_{x_0}$$

$$y = \frac{br_{\hat{x}_0} + pc_{x_0}}{ap + bq}$$

$$y = \frac{b}{ap + bq}r_{\hat{x}_0} + \frac{p}{ap + bq}c_{x_0}$$

$$u = -\frac{q}{p}y + \frac{r_{\hat{x}_0}}{p}$$

$$u = -\frac{bq}{ap}u - \frac{qc_{x_0}}{ap} + \frac{ar_{\hat{x}_0}}{ap}$$

$$(ap + bq)u = -qc_{x_0} + ar_{\hat{x}_0}$$

$$u = \frac{ar_{\hat{x}_0} - qc_{x_0}}{ap + bq}$$

$$u = \frac{a}{ap + bq}r_{\hat{x}_0} - \frac{q}{ap + bq}c_{x_0}$$

closed-loop characteristic polynomial

# Feedback systems

When considered as a two-output system

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \dfrac{p}{ap+bq} & \dfrac{b}{ap+bq} \\ \dfrac{-q}{ap+bq} & \dfrac{a}{ap+bq} \end{bmatrix} \begin{bmatrix} c_{x_0} \\ r_{\hat{x}_0} \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} a & -b \\ q & p \end{bmatrix}^{-1} \begin{bmatrix} c_{x_0} \\ r_{\hat{x}_0} \end{bmatrix}$$

# Closed-loop characteristic polynomial

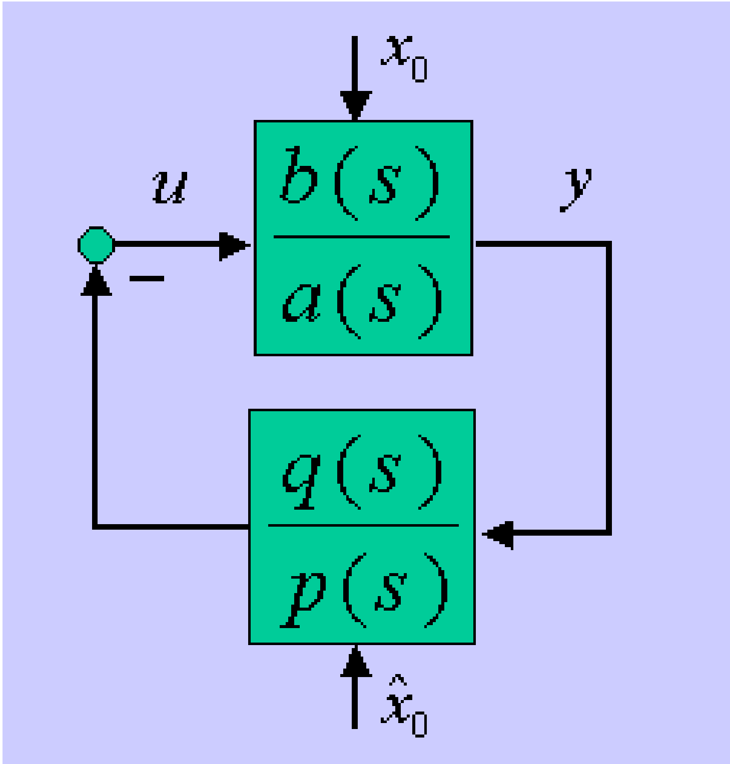
If there are no hidden modes in the plant and controller descriptions, then

$$a(s)p(s)+b(s)q(s)$$

is the

characteristic polynomial

of the closed loop



# Properness of the closed loop

(When working in $s$ or $z$), we must also take care of the overall system properness!

Equivalent expressions:

- All the transfer functions before are proper
- The closed loop system is internally proper
- There are no strange algebraic loops
- In state space: $\det(I+D_1D_2)$ is nonzero
- RHS degree = plant order + controller order

This is result guaranteed if

- either plant is strictly proper and RHS degree is “high enough” (see further)
- If it is not, then we must be careful and check the result
- Either plant or controller is strictly proper for d-t systems

# Comments on degrees

[ PolyX ]

## Comments

- What is the “large enough” degree of RHS?
  Generically: If the plant order is $n$=deg a, then the (min. deg. $q$) controller is at least of order $n$-1 (as deg $q$ = $n$-1), hence the overall system is at least of order 2$n$-1 and so is degree of the RHS.
- In contrast to the design in $d$, every choice of $m$ leads to a unique solution. Explain the difference!
- If plant is strictly proper, then the (min. deg. of $q$) solution is always proper
  (as then deg($bq$)<deg($ap$)=deg RHS )!
- If plant is proper but not strictly, then the solution is usually proper but not always. If it is not, then the only remedy is to take a RHS with degree >2$n$-1 !

# Feedback design

///

# Pole placement

## Pole placement (*s* or *z*)

this is new

Given a polynomial

$$m(d) = (d - d_1)\cdots(d - d_{2n-1})$$

of degree at least 2deg *a* – 1, solve

$$ap + bq = m$$

and take the solution with minimum degree of *q*.

Solvability condition $(a, b) \mid m$

Plant and controller free of hidden modes!

Eventual uncontrollable modes must be preserved.
Of course, they must be stable.

# Comments on degrees

## Comments

- What is the “large enough” degree of $m$ ? Generically: If the plant order is $n$=deg a, then the (min. deg. of $q$) controller is at least of order $n$-1
  (as deg $q = n$-1), hence the overall system is at least of order 2$n$-1 and so is deg $m$.
- In contrast to the design in $d$, every choice of $m$ leads to a unique solution. Explain the difference!
- If plant is strictly proper, then the (min. deg. of $q$) solution is always proper
  (as then deg($bq$)<deg($ap$)=deg $m$ )!
- If plant is proper but not strictly, then the solution is usually proper but not always. If it is not, then the only remedy is to take $m$ with degree >2$n$-1 !

# Stabilization

## Stabilization (in *s* and *z*)

### Solution

To stabilize the plant take any stable polynomial *e* of degree at least 2deg *a* – 1 and solve the equation

$$ap + bq = e$$

implicit parameterization

the controller results from the solution with minimum degree of *q*.

### Solvability

The plant free of unstable hidden modes and with $(a, b)$ stable. That is, its uncontrollable part must be stable.

### Properness

Guaranteed is plant is strictly proper, must be checked otherwise.

# All stabilizing controllers: Youla-Kucera

Assumption: $(a,b)$ is coprime. If not, make it so and proceed as:

All stabilizing controllers

If $f$ is a stable polynomial of degree $\deg a$ , if $g$ is another stable polynomial of degree $\deg a - 1$ and if $x,y$ is the solution of

$$ax + by = fg$$

Then all stabilizing controllers are parameterized by

$$\frac{q}{p} = \frac{\dfrac{y}{g} + \dfrac{a}{f}\dfrac{t}{h}}{\dfrac{x}{g} - \dfrac{b}{f}\dfrac{t}{h}}$$

where $t$ is arbitrary polynomial and $h$ is arbitrary stable but such that $t/h$ is proper and $p$ is nonzero.

# Some details

- for $t=0$, we have $p=x$, $q=y$ and the c-l characteristic polynomial equals $m=fg$ .
- for other $t$, we have

$$\frac{q}{p} = \frac{fhy + gat}{fhx - gbt}$$

where we cancel as much as we can and the resulting c-l characteristic polynomial seems to equal $m=f^2hg$ but usually some factors disappear due to the cancellation in controller.

- remember that $t/h$ must be proper .
- If plant is strictly proper, the solution is proper.

# More and more stable?

- nothing like most stable system exist in c-t
- the further to left are the desired poles located, the faster system reacts,
- but this has no limit
- in contrast to the discrete-time case

# Asymptotic regulation

## Asymptotic regulation

### Formulation

Achieved iff both sequences $y(s), u(s)$ are Hurwitz stable and proper for any combination of $c_{x_0}, r_{\hat{x}_0}$ .

### Solution

All asymptotic regulators result from the solution (with min deg q) of

$$ap + bq = m$$

for a stable polynomial *m* of degree 2deg *a* – 1 and higher .

### Solvability condition

$(a, b)$ stable

AR is equivalent to stabilization !

# Deadbeat ?

- nothing like that exists in linear c-t systems
- explain why is a d-t deadbeat controller can work for a c-t plant?

# Tracking problems

## Asymptotic tracking

### Formulation

Achieved iff both sequences

$u, e = y_r - y$ are Schur stable

for any combination of $c_{x_0}, s_{\hat{x}_0}, g_{\tilde{x}_0}$

### Solution

min. deg. $q$ and min. deg $r$

All asymptotic regulators result from the solution of two equations

$ap + bq = m$ and $f^- t + br = m$

for a stable polynomial $m$ of degree at least 2deg *a* -1.

### Solvability

deg *r* < deg *p*

1) $(a, b)$ stable; 2) $(f^-, b) = 1$ ; 3) $f^- \mid a$ .

# MIMO systems

# SS and IO – MIMO case

## Computation of the IO representation

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

$$y = Cx + Du$$

$$y(s) = \left(C(sI - A)^{-1}B + D\right)u(s) + C(sI - A)^{-1}x_0$$

$$y(s) = P^{-1}(s)Q(s)\,u(s) + P^{-1}(s)R_{x_0}(s)$$

$$Q_1(s)P_1^{-1}(s)$$

$$\det(sI - A) = \det P(s) = \det P_1(s)$$

$$i(sI - A) = i(P(s)) = i(P_1(s))$$

# Feedback – MIMO case

- if plant $A^{-1}(s)B(s)$ and controller $Q(s)P^{-1}(s)$ then the "feedback equation"

$$A(s)P(s) + B(s)Q(s) = M(s)$$

- if plant $B(s)A^{-1}(s)$ and controller $P^{-1}(s)Q(s)$ then the "feedback equation"

$$P(s)A(s) + Q(s)B(s) = M(s)$$

# Particular case: state space problems

- state feedback: $(sI - A)^{-1}B$ and controller

$$L = Q(s)P^{-1}(s)$$

$$(sI - A)P(s) + BQ(s) = M(s)$$

- output injection: $C(sI - A)^{-1}$ and injector

$$K = P^{-1}(s)Q(s)$$

$$P(s)(sI - A) + Q(s)C = M(s)$$

# Analysis

- check *M* for stability, zeros (=c-l poles), invariant factors !!!

# Design – sketched in *d*

- deadbeat: either

$$A(d)P(d)+B(d)Q(d)=I$$

or

$$P(d)A(d)+Q(d)B(d)=I$$

- invariant factors assignment (pole placement)

$$A(d)P(d)+B(d)Q(d)=M(d)$$

$$P(d)A(d)+Q(d)B(d)=M(d)$$

- stabilization

- in *z* and *s* : setting degrees more involved

# Linear quadratic regulator

# Linear quadratic regulator–1

State-feedback:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ u(t) &= -Lx(t)\end{aligned}$$

Regulated variable:

$$z(t) = C_R x(t) + J_R u(t)$$

## LQ regulator problem:

Find control function $u(t), t \geq 0$ which minimizes $\|z\|_2$ for every initial state $x(0)$ .

# Linear quadratic regulator–2

State-space solution: (assumed: $J_R^T C_R = 0, J_R^T J_R = I$ )

If $(A, B)$ stabilizable, a unique optimal control exists

$$u(t) = -Lx(t), \;\; L = G^T P$$

Where $P$ is the least symmetric non-negative definite solution of:

$$PA + A^T P - PBB^T P + C_R^T C_R = 0$$

ARE

If $(A, C_R)$ detectable, $\dot{x}(t) = (A - BL)x(t)$ is stable.

# Linear quadratic regulator–3

Polynomial solution:

$$(sI - A)^{-1} B = N_R(s) D_R^{-1}(s)$$

l2r conversion

right coprime

spectral factorization

$$F^T(-s)F(s) = N_R^T(-s) C_R^T C_R N_R(s) + D_R^T(-s) D_R(s)$$

stable

$$X_L D_R(s) + Y_L N_R(s) = F(s)$$

linear equation

constant solution

$$L = X_L^{-1} Y_L$$

state feedback

# Linear Gaussian filter

# Linear Gaussian filter–1

Linear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_L u(t) \\ y(t) &= Cx(t) + J_L v(t) \end{aligned}$$

Gaussian white random process with zero mean and unity covariance: $v(t)$

## LQ filter problem:

Determine the conditional mean estimate $\hat{x}(t)$ of the current state $x(t)$ given the past observations $y(\tau), \tau \le t$

# Linear Gaussian filter–2

State-space solution: (for simplicity assumed: $G_L J_L^T = 0, J_L J_L^T = I$ )

If $(A, C)$ is detectable, the estimate is generated by the filter

$$\dot{\hat{x}}(t) = A\hat{x}(t) + QC^T\left[y(t) - C\hat{x}(t)\right]$$

Where $P$ is the least symmetric non-negative definite solution of:

$$AQ + QA^T - QB^T BQ + B_L B_L^T = 0$$

ARE

The filter is obtained from the original system by applying an input injection

$$K = -QC^T$$

If $(A, B_L)$ detectable, the filter is stable.

# Linear Gaussian filter–3

Polynomial solution:

r2l conversion

$$C(sI - A)^{-1} = D_L^{-1}(s)N_L(s)$$

left coprime

spectral factorization

$$G(s)G^T(-s) = N_L(s)B_L B_L^T N_R^T(-s) + D_L(s)D_L^T(-s)$$

stable

$$A_L(s)X_R + B_L(s)Y_R = G(s)$$

constant solution

linear equation

$$K = Y_R X_R^{-1}$$

output injection

# Spectral factorization

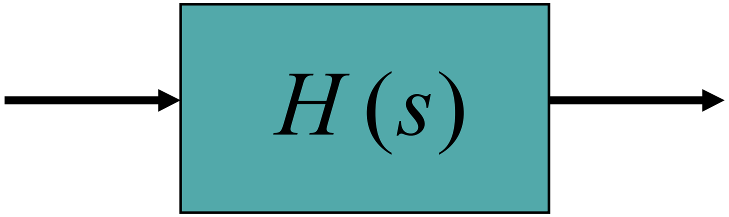
[ PolyX ]

# Spectral factorization - 1

input spectral density

$$\Phi_u = I$$

vector white random process




stable

output spectral density $\Phi_Y(s)$

$$\Phi_Y(s) = H(s)H^*(s)$$

where

conjugation

$$H^*(s) = H^T(-s)$$

- para-Hermitian symmetric $\Phi_Y(s) = \Phi_Y^*(s)$
- positive definite on imaginary axis $\Phi_Y(j\omega) > 0 \quad \forall \omega \in \Re$

# Spectral factorization - 2

## Conversed problem:

**Given output spectral density, construct the (stable) system**

Given rational matrix $\Phi_Y(s)$ such that

- para-Hermitian symmetric $\Phi_Y(s) = \Phi_Y^{*}(s)$
- positive definite on Im $\Phi_Y(j\omega) > 0 \quad \forall \omega \in \Re$

Find stable rational matrix $H(s)$ such that

**Rational spectral factorization**

$$\Phi_Y(s) = H(s)H^{*}(s)$$

# Spectral factorization - 3

## Polynomial spectral factorization

Given polynomial matrix $M(s)$ that is

- para-Hermitian symmetric $M(s) = M^*(s)$
- positive (nonnegative) definite on Im

Find stable polynomial matrix $X(s)$ such that

$$M(s) = X(s)X^*(s)$$

Quadratic problem

factorization

Why spectral?

# Spectral factorization - 4

Example $a(s) = \underbrace{\left(2 - 2s + s^2\right)}_{1 \pm j}\underbrace{\left(2 + s\right)}_{-2}\underbrace{\left(3 - s\right)}_{3}$

$$a^*(s) = a(-s) = \underbrace{\left(2 + 2s + s^2\right)}_{-1 \pm j}\underbrace{\left(2 - s\right)}_{2}\underbrace{\left(3 + s\right)}_{-3}$$

$$m(s) = a(s)a^*(s) \quad \Rightarrow \quad m(s) = x(s)x^*(s)$$

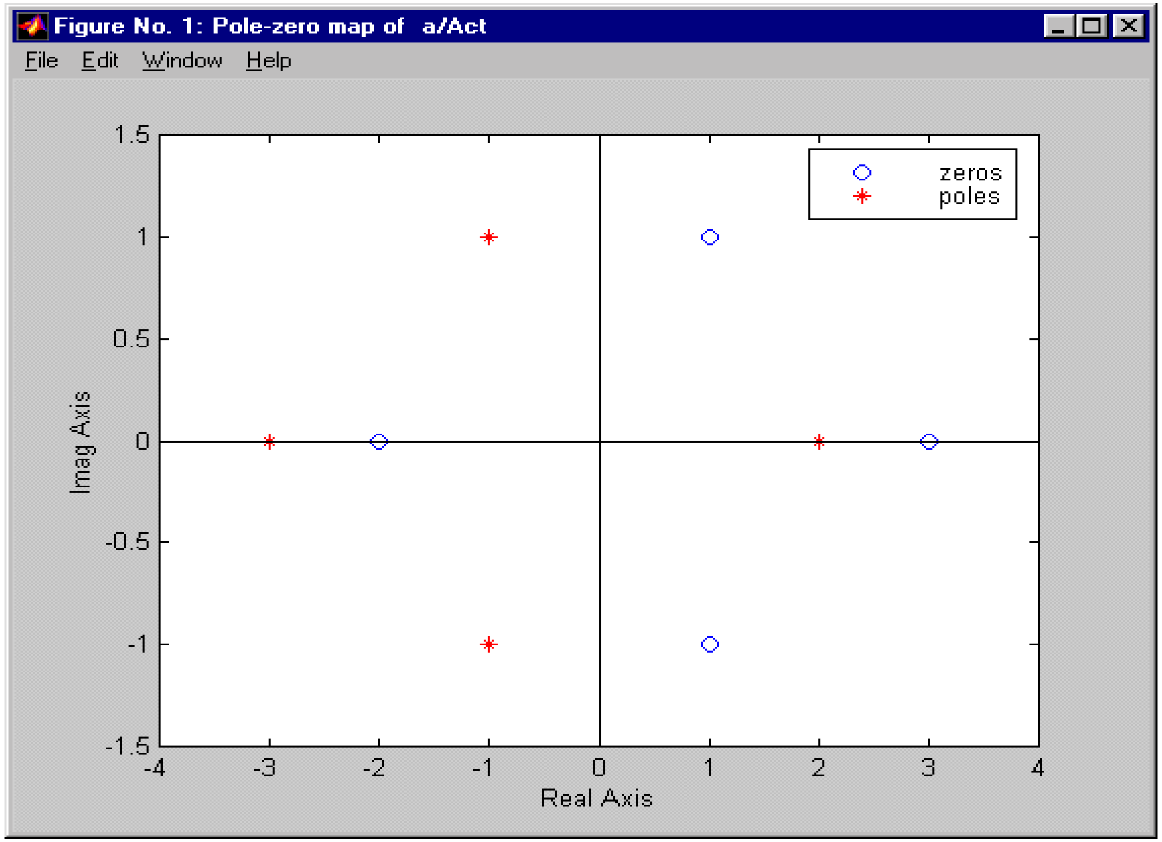
by inspection

$$x(s) = \underbrace{\left(2 + 2s + s^2\right)}_{-1 \pm j}\underbrace{\left(2 + s\right)}_{-2}\underbrace{\left(3 + s\right)}_{-3}$$

# Spectral factorization - 5

Example

```
a=(2-2*s+s^2)*(2+s)*(3-s);
pzplot(a,a')
```



# Spectral factorization - 6

## Canonical factorization:

$M(s)$ diagonally reduced

$X(s)$ row reduced

## Uniqueness:

$$X'(s) = X(s)U$$

$$UU^T = I$$

Constant, orthogonal

# Spectral factorization - 6

## Polynomial J-spectral factorization

more general

Given polynomial matrix $M(s)$ that is

- para-Hermitian symmetric $M(s) = M^*(s)$
- if zeros on Im, then of even multiplicity in each inv. pol.

Find stable polynomial matrix $X(s)$ such that

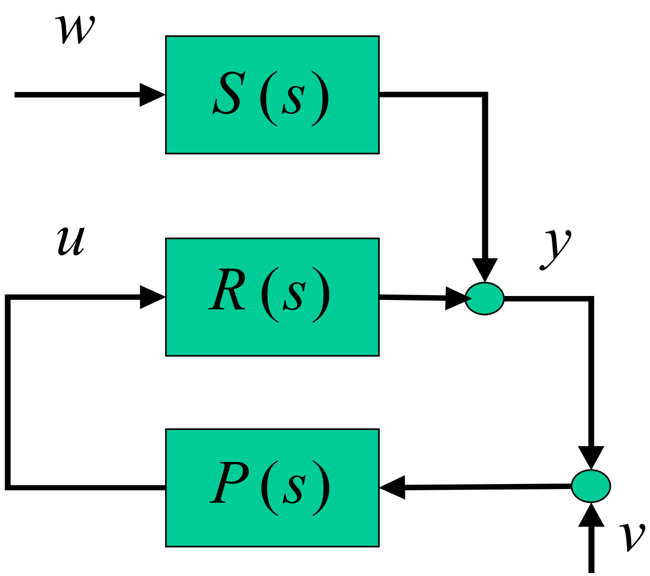
$$M(s) = X(s)JX^*(s)$$

$$J = \mathrm{diag}\{+1,\ldots,-1,\ldots,0,\ldots\}$$

Signature matrix

# LQG

# LQG - 1




$$y(s) = R(s)u(s) + S(s)w(s)$$

$$u(s) = -P(s)\left(y(s) - v(s)\right)$$

background noise $w(s)\ldots Q_1$

observation noise $v(s)\ldots R_1$

(independent, zero-mean, stationary, white, Gaussian)

$$\lim_{t\to\infty} E\left\{u^T(t)R_2u(t) + y^T(t)Q_2y(t)\right\}$$

# LQG - 2

## Polynomial solution

Given

- left coprime lmf

$$A^{-1}\begin{bmatrix} B & C \end{bmatrix} = \begin{bmatrix} R & S \end{bmatrix}$$

- coprime pmf's

$$A_0^{-1} B_0 = B_1 A_1^{-1} = R$$

Compute

- 2 spectral factorizations

$$A R_1 A^* + C Q_1 C^* = G G^*$$

noise

$$A_1^* R_2 A_1 + B_1^* Q_2 B_1 = F^* F$$

cost

# LQG - 3

- 2 coprime pmf's

$$B_2 G_1^{-1} = G^{-1} B$$
$$A_2 G_2^{-1} = G^{-1} A$$

- linear matrix polynomial equation (two-sided)

$$F^*[X \ \ Y] - Z^*\begin{bmatrix} B_2 & -A_2 \end{bmatrix} = \begin{bmatrix} A_1^* R_2 G_1 & B_1^* Q_2 G_2 \end{bmatrix}$$

- special solution (via division)

$X_1, Y_1, Z_1$ such that $F^{-1} Z_1$ strictly proper

# LQG - 4

- coprime lmf

$$G_0^{-1}\begin{bmatrix} X_0 & Y_0 \end{bmatrix} = \begin{bmatrix} X_1 G_1^{-1} & Y_1 G_2^{-1} \end{bmatrix}$$

- controller

$$P = X_0^{-1} Y_0$$

## Solvability conditions

- stable greatest common left divisor of $A, B$
- (4 rational matrices strictly proper)
- nonsingular $X_0$

# LQG - 5

[PolyX]

$$F^*\begin{bmatrix} X_1 & Y_1 \end{bmatrix} - Z^*\begin{bmatrix} B_2 & -A_2 \end{bmatrix} = \begin{bmatrix} A_1^* R_2 G_1 & B_1^* Q_2 G_2 \end{bmatrix} \times \begin{bmatrix} G_1^{-1} A_1 \\ G_2^{-1} B_1 \end{bmatrix}$$

$$F^*\begin{bmatrix} X_1 G_1^{-1} A_1 & Y_1 G_2^{-1} B_1 \end{bmatrix} - Z^* G^{-1}\begin{bmatrix} B A_1 - A B_1 \end{bmatrix} =$$

$$= A_1^* R_2 A_1 + B_1^* Q_2 B_1 = F^* F$$

$$X_1 G_1^{-1} A_1 + Y_1 G_2^{-1} B_1 = F$$

pole placement

$$X_0 A_1 + Y_0 B_1 = G_0 F$$

implied equation

# LQG - 6

simpler solution for $C = B$ $\quad A_0^{-1} B_0 = B_1 A_1^{-1} = R$

$$X_1 A_1 + Y_1 B_1 = F$$

Polynomial part $\quad T_1 = \Pi\left(Y_1 A_0^{-1} G\right)$

$$T_1 G_1^{-1} = \overline{G}^{-1} \overline{T}^{-1}$$

$$X_0 = \overline{G} X_1 + \overline{T}_1 B_0$$

$$Y_0 = \overline{G} Y_1 - \overline{T}_1 A_0$$

$$\Longrightarrow \quad X_0 A_1 + Y_0 B_1 = \overline{G} F$$

# Comparison with state-space solution

# Comparison - 1

$$J = E\left(\int_{-\infty}^{\infty}[x^T(t)Q_2x(t)+u^T(t)R_2u(t)]dt\right)$$

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$

$$y(t) = Cx(t) + v(t)$$

$w, v$ Independent, white, Gaussian, zero means
intensities $Q_1 \geq 0,\ R_1 > 0$

$Q_2 \geq 0,\ R_2 > 0$ weighting matrices

# Comparison - 2

State space solution - Separation theorem

Optimal control law $u(t) = -L_2 \hat{x}(t)$

$\hat{x}(t)$ is estimate of $x(t)$ generated by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L_1(t)\left(y(t) - C\hat{x}(t)\right)$$

$$L_1 = P_1 C^T R_1^{-1} \qquad L_2 = R_2^{-1} B^T P_2$$

$$A P_1 + P_1 A^T - P_1 C^T R_1^{-1} C^T P_1 + Q_1 = 0$$

$$P_2 A + A^T P_2 - P_2 B R_2^{-1} B^T P_2 + Q_2 = 0$$

# Comparison - 3

$(C, A)$ detectable

$\begin{bmatrix} -A^T & C^T R_1^{-1} C \\ Q_1 & A \end{bmatrix}$ no purely imaginary eigenvalue

⇒ estimator exists and is stable

$(A, B)$ stabilizable

$\begin{bmatrix} -A & B R_2^{-1} B^T \\ Q_2 & A^T \end{bmatrix}$ no purely imaginary eigenvalue

⇒ regulator exists and is stable

# Comparison - 4

If stabilizing $P_1$ exists but the pair $(A, Q_1)$ is not stabilizable, then optimal estimator is not stable.

Take best stable estimator ( $P_1$ )

If stabilizing $P_2$ exists but the pair $(Q_2, A)$ is not detectable, then optimal estimator is not stable.

Take best stable regulator( $P_2$ )

What if $P_1$ or $P_2$ does not exist?

If no stable estimator and/or regulator?

# Comparison - 5

## Polynomial solution

$$\left(sI - A\right)^{-1} B = \frac{\bar{b}(s)}{a(s)}, \quad C\left(sI - A\right)^{-1} = \frac{\bar{c}(s)}{a(s)}$$

$$C\left(sI - A\right)^{-1} B = \frac{b(s)}{a(s)}$$

$(A, B)$ controllable

$(C, A)$ observable

# Comparison - 6

$$a(s)R_1a^*(s)+\bar{c}(s)Q_1\bar{c}^*(s)=f(s)f^*(s)$$

$$a(s)R_2a^*(s)+\bar{b}(s)Q_2\bar{b}^*(s)=g^*(s)g(s)$$

$$a(s)x(s)+b(s)y(s)=f(s)g(s)$$

$$\deg y(s)<\deg a(s)$$

$$-\frac{y(s)}{x(s)}$$

# Comparison - 7

May work even if $(Q_2, A)$ is undetectable and/or $(A, Q_1)$ is unstabilizable and the corresponding eigenvalue is purely imaginary.

# Comparison - 7

Example:

$$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, B=\begin{bmatrix}0\\ 1\end{bmatrix}, C=\begin{bmatrix}1 & 1\end{bmatrix}$$

$$Q_1=\begin{bmatrix}16 & 0\\ 0 & 0\end{bmatrix}, Q_2=\begin{bmatrix}16 & 0\\ 0 & 1\end{bmatrix}$$

$$R_1=1, R_2=1$$

$(A,Q_1)$ is not stabilizable and

$\begin{bmatrix}-A^T & C^T R_1^{-1} C\\ Q_1 & A\end{bmatrix}$ has a zero eig. ⇒ **Stable estimator does not exits.**

# Comparison - 8

$$Q_1 = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a(s) = s^2, b(s) = 1 + 4s + 6s^2$$

$$\bar{c}(s) = \begin{bmatrix} s^2 & s + 4s^2 & 1 + 4s + 6s^2 \end{bmatrix}, \bar{b}(s) = \begin{bmatrix} 1 \\ s \end{bmatrix}$$

$$f(s) = s\left(4 + s\right), g(s) = 4 + 3s + s^2$$

# Comparison - 9

$$x(s) = s\left(7 + s\right), \quad y(s) = 16s$$

$$-\frac{y(s)}{x(s)} = -\frac{16}{7+s}$$

The controller has an unobservable mode. When minimally realize, its is the optimal controller.

The unobservable mode is not necessary for control and hence need **not be estimated !!!**

# $H_2$ Optimization

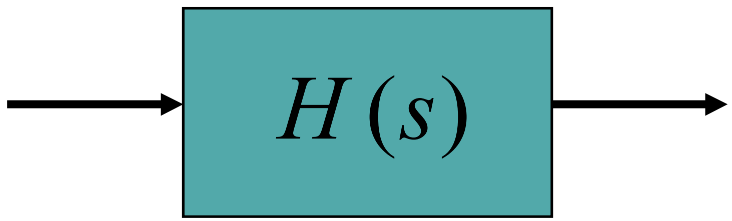
[ PolyX ]

# $H_2$ Optimization- 1

input spectral density

$\Phi_u(s) = I$

vector white random process




stable

output spectral density $\Phi_y(s)$

$$\Phi_Y(s) = H(s)H^*(s)$$

$H_2$ optimization = removing the stochastic interpretation from LQG

# $H_2$ Optimization- 2

Variance of the output signal

$$Ey^T(t)y(t) = tr\,Ey(t)y^T(t)$$

$$= tr\int_{-\infty}^{\infty} H(j\omega)H^T(-j\omega)\,df$$

**2-norm $\|H\|_2$ of system transfer matrix**

$$f = \frac{\omega}{2\pi}$$