

Polynomial Methods for Control Analysis and Design

[PolyX]

5/ Continuous-time
and
MIMO systems

Overview

[PolyX]

Ch. 1. Polynomials and polynomial matrices

Ch. 2. Polynomial toolbox

computer session

Ch. 3. Polynomials in control systems

Ch. 4. Discrete-time systems

Ch. 5. Continuous-time and MIMO systems

Ch. 6. CAD based on polynomial methods

computer session

Ch. 7. Future perspectives

Overview

[PolyX]

Ch. 5. Continuous-time and MIMO systems

C-t systems and polynomials

Feedback systems

Feedback design

Tracking problems

MIMO systems

Linear quadratic regulator

Linear Gaussian filter

Spectral factorization

LQG

H_2 optimization

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Continuous-time systems and polynomials

[PolyX]

???

SS and IO representation

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Computation of the IO representation

$$\dot{x} = Ax + Bu \quad x(0) = x_0$$

$$y = Cx + Du$$

Using Laplace transform $\mathcal{L}(\dot{x}) = sx(s) - x(0) = sx(s) - x_0$

$$y(s) = \left(C(sI - A)^{-1}B + D \right) u(s) + C(sI - A)^{-1}x_0$$

$$y(s) = \frac{q(s)}{p(s)} u(s) + \frac{r_{x_0}(s)}{p(s)}$$

r_{x_0} is not used in design

$$p(s) = \det(sI - A)$$

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Hidden modes

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Hidden modes:

Let $p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$ is IO model of $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

that is $\frac{q(s)}{p(s)} = \left(C(sI - A)^{-1}B + D \right)$, $\frac{r_{x_0}(s)}{p(s)} = C(sI - A)^{-1}x_0$

We say that it has no hidden modes iff

$$p(s) = \det(sI - A)$$

Equivalent expressions:

- Nothing has been cancelled during computation
- "What you see is what you have"

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Convention

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It is always assumed that there is
no common factor present (at the same time)
in all the polynomials p, q, r_{x_0}

Remember that r_{x_0} is a "set" !

Then no hidden modes requirement means

- the system is observable
- the system is controllable iff p and q are coprime

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Plant

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Plant

The given system to be controlled

- No hidden modes
- Don't care about c_{x_0}

$$y(s) = \frac{b(s)}{a(s)}u(s) + \frac{c_{x_0}(s)}{a(s)}$$

New assumption:
plant is proper, i.e.
both fractions are proper

identification

$$C(sI - A)^{-1}B + D$$

$$C(sI - A)^{-1}x_0$$

$$C(sE - A)^{-1}B + D$$

$$C(sE - A)^{-1}x_0$$

$$\dot{x} = Ax + Bu, x(0) = x_0$$

$$y = Cx + Du$$

$$E\dot{x} = Ax + Bu, x(0) = x_0$$

$$y = Cx + Du$$

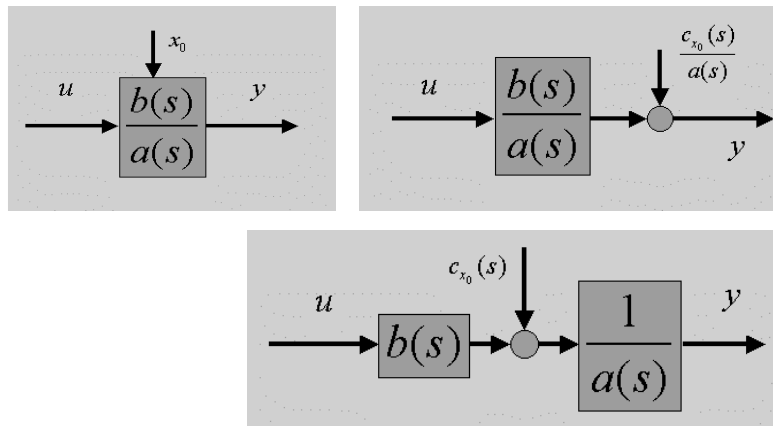
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Plant

Some equivalent pictures



Controller

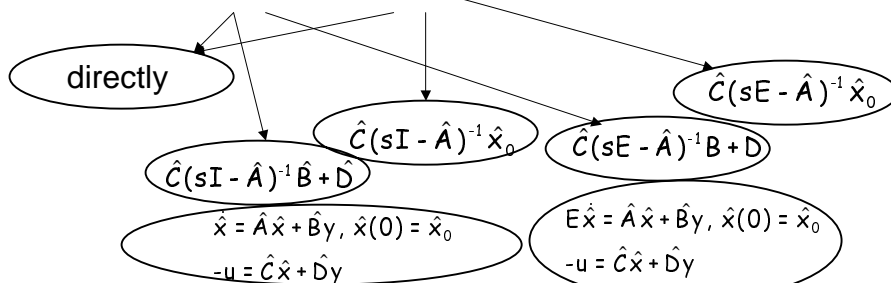
Controller

The system to be found

- To meet design specs.
- Realized without hidden modes
- r_{x_0} can be any

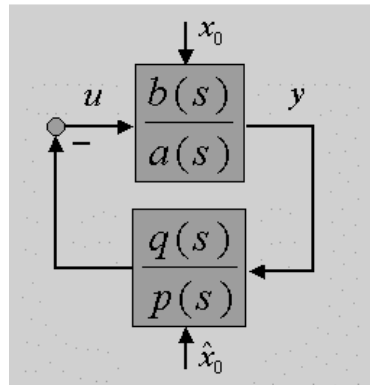
$$u(s) = \frac{q(s)}{p(s)} y(s) + \frac{r_{x_0}(s)}{p(s)}$$

New requirement:
controller is proper =
both fractions are proper



Feedback systems

Consider a simple feedback system



Plant

$$y(s) = \frac{b(s)}{a(s)}u(s) + \frac{c_{x_0}(s)}{a(s)}$$

Controller

$$u(s) = -\frac{q(s)}{p(s)}y(s) + \frac{r_{\hat{x}_0}(s)}{p(s)}$$

Important assumption: No hidden modes !

Feedback systems

$$y = \frac{b}{a}u + \frac{c_{x_0}}{a}$$

$$y = -\frac{bq}{ap}y + \frac{br_{\hat{x}_0}}{ap} + \frac{pc_{x_0}}{ap}$$

$$(ap + bq)y = br_{\hat{x}_0} + pc_{x_0}$$

$$y = \frac{br_{\hat{x}_0} + pc_{x_0}}{ap + bq}$$

$$y = \frac{b}{\underbrace{ap + bq}}r_{\hat{x}_0} + \frac{p}{\underbrace{ap + bq}}c_{x_0}$$

$$u = -\frac{q}{p}y + \frac{r_{\hat{x}_0}}{p}$$

$$u = -\frac{bq}{ap}u - \frac{qc_{x_0}}{ap} + \frac{ar_{\hat{x}_0}}{ap}$$

$$(ap + bq)u = -qc_{x_0} + ar_{\hat{x}_0}$$

$$u = \frac{ar_{\hat{x}_0} - qc_{x_0}}{ap + bq}$$

$$u = \frac{a}{\underbrace{ap + bq}}r_{\hat{x}_0} - \frac{q}{\underbrace{ap + bq}}c_{x_0}$$

closed-loop characteristic polynomial

Feedback systems

PolyX

When considered as a two-output system

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \frac{p}{ap+bq} & \frac{b}{ap+bq} \\ \frac{-q}{ap+bq} & \frac{a}{ap+bq} \end{bmatrix} \begin{bmatrix} c_{x_0} \\ r_{\hat{x}_0} \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} a & -b \\ q & p \end{bmatrix}^{-1} \begin{bmatrix} c_{x_0} \\ r_{\hat{x}_0} \end{bmatrix}$$

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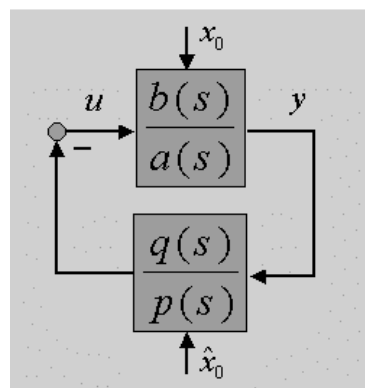
Closed-loop characteristic polynomial

PolyX

If there are no hidden modes in the plant and controller descriptions, then

$$a(s)p(s) + b(s)q(s)$$

is the
characteristic polynomial
of the closed loop



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Properness of the closed loop



(When working in s or z), we must also take care of the overall system properness!

Equivalent expressions:

- All the transfer functions before are proper
- The closed loop system is internally proper
- There are no strange algebraic loops
- In state space: $\det(I+D_1D_2)$ is nonzero
- RHS degree = plant order + controller order

This result guaranteed if

- either plant is strictly proper and RHS degree is “high enough” (see further)
- If it is not, then we must be careful and check the result
- Either plant or controller is strictly proper for d-t systems

Comments on degrees



Comments

- What is the “large enough” degree of RHS?
Generically: If the plant order is $n=\deg a$, then the (min. deg. q) controller is at least of order $n-1$ (as $\deg q = n-1$), hence the overall system is at least of order $2n-1$ and so is degree of the RHS.
- In contrast to the design in d , every choice of m leads to a unique solution. Explain the difference!
- If plant is strictly proper, then the (min. deg. of q) solution is always proper
(as then $\deg(bq) < \deg(ap) = \deg \text{RHS}$)!
- If plant is proper but not strictly, then the solution is usually proper but not always. If it is not, then the only remedy is to take a RHS with degree $>2n-1$!

Feedback design



PolyX

///

Pole placement

Pole placement (s or z)

this is new

Given a polynomial

$$m(d) = (d - d_1) \cdots (d - d_{2n-1})$$

of degree at least $2\deg a - 1$, solve

$$ap + bq = m$$

and take the solution with minimum degree of q .

Solvability condition

$$(a, b) \mid m$$

Plant and controller
free of hidden modes!

Eventual uncontrollable modes must be preserved.
Of course, they must be stable.

Comments on degrees

PolyX

Comments

- What is the “large enough” degree of m ? Generically: If the plant order is $n = \deg a$, then the (min. deg. of q) controller is at least of order $n-1$ (as $\deg q = n-1$), hence the overall system is at least of order $2n-1$ and so is $\deg m$.
- In contrast to the design in d , every choice of m leads to a unique solution. Explain the difference!
- If plant is strictly proper, then the (min. deg. of q) solution is always proper (as then $\deg(bq) < \deg(ap) = \deg m$)!
- If plant is proper but not strictly, then the solution is usually proper but not always. If it is not, then the only remedy is to take m with degree $> 2n-1$!

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Stabilization

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Stabilization (in s and z)

Solution

To stabilize the plant take any stable polynomial e of degree at least $2\deg a - 1$ and solve the equation

$$ap + bq = e$$

implicit parameterization

the controller results from the solution with minimum degree of q .

Solvability

The plant free of unstable hidden modes and with (a, b) stable. That is, its uncontrollable part must be stable.

Properness

Guaranteed is plant is strictly proper, must be checked otherwise.

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All stabilizing controllers: Youla-Kucera

PolyX

Assumption: (a,b) is coprime. If not, make it so and proceed as:

All stabilizing controllers

If f is a stable polynomial of degree $\deg a$, if g is another stable polynomial of degree $\deg a - 1$ and if x,y is the solution of

$$ax + by = fg$$

Then all stabilizing controllers are parameterized by

$$\frac{q}{p} = \frac{\frac{y}{g} + \frac{a}{f} \frac{t}{h}}{\frac{x}{g} - \frac{b}{f} \frac{t}{h}}$$

where t is arbitrary polynomial and h is arbitrary stable but such that t/h is proper and p is nonzero.

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Some details

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- for $t=0$, we have $p=x, q=y$ and the c-l characteristic polynomial equals $m=fg$.
- for other t , we have

$$\frac{q}{p} = \frac{fhy + gat}{fhx - gbt}$$

where we cancel as much as we can and the resulting c-l characteristic polynomial seems to equal $m=f^2hg$ but usually some factors disappear due to the cancellation in controller.

- remember that t/h must be proper.
- If plant is strictly proper, the solution is proper.

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More and more stable?

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- nothing like most stable system exist in c-t
- the further to left are the desired poles located, the faster system reacts,
- but this has no limit
- in contrast to the discrete-time case

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Asymptotic regulation

PolyX

Asymptotic regulation

Formulation

Achieved iff both sequences $y(s), u(s)$ are Hurwitz stable and proper for any combination of c_{x_0}, r_{x_0} .

Solution

All asymptotic regulators result from the solution (with min deg q) of

$$ap + bq = m$$

for a stable polynomial m of degree $2\deg a - 1$ and higher.

Solvability condition

(a, b) stable

AR is equivalent to stabilization !

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Deadbeat ?



- nothing like that exists in linear c-t systems
- explain why is a d-t deadbeat controller can work for a c-t plant?

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Tracking problems



Asymptotic tracking

[PolyX]

Asymptotic tracking

Formulation

Achieved iff both sequences

$u, e = y_r - y$ are Schur stable

for any combination of $c_{\hat{x}_0}, s_{\hat{x}_0}, g_{\hat{x}_0}$

Solution

min. deg. q and min. deg r

All asymptotic regulators result from the solution of two equations

$$ap + bq = m \quad \text{and} \quad f^-t + br = m$$

for a stable polynomial m of degree at least $2\deg a - 1$.

Solvability

deg $r < \deg p$

1) (a, b) stable; 2) $(f^-, b) = 1$; 3) $f^- | a$.

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MIMO systems

[PolyX]

SS and IO – MIMO case

PolyX

Computation of the IO representation

$$\dot{x} = Ax + Bu \quad x(0) = x_0$$

$$y = Cx + Du$$

$$y(s) = \left(C(sI - A)^{-1}B + D \right) u(s) + C(sI - A)^{-1}x_0$$

$$y(s) = P^{-1}(s)Q(s) u(s) + P^{-1}(s)R_{x_0}(s)$$

$$Q_1(s)P_1^{-1}(s)$$

$$\det(sI - A) = \det P(s) = \det P_1(s)$$

$$i(sI - A) = i(P(s)) = i(P_1(s))$$

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Feedback – MIMO case

PolyX

- if plant $A^{-1}(s)B(s)$ and controller $Q(s)P^{-1}(s)$ then the “feedback equation”

$$A(s)P(s) + B(s)Q(s) = M(s)$$

- if plant $B(s)A^{-1}(s)$ and controller $P^{-1}(s)Q(s)$ then the “feedback equation”

$$P(s)A(s) + Q(s)B(s) = M(s)$$

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Particular case: state space problems

PolyX

- state feedback: $(sI - A)^{-1}B$ and controller

$$L = Q(s)P^{-1}(s)$$

$$(sI - A)P(s) + BQ(s) = M(s)$$

- output injection: $C(sI - A)^{-1}$ and injector

$$K = P^{-1}(s)Q(s)$$

$$P(s)(sI - A) + Q(s)C = M(s)$$

Analysis

PolyX

- check M for stability, zeros (=c-l poles), invariant factors !!!

Design – sketched in d

[PolyX]

- deadbeat: either $A(d)P(d) + B(d)Q(d) = I$
or $P(d)A(d) + Q(d)B(d) = I$
- invariant factors assignment (pole placement)
 $A(d)P(d) + B(d)Q(d) = M(d)$
 $P(d)A(d) + Q(d)B(d) = M(d)$
- stabilization
- in z and s : setting degrees more involved

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Linear quadratic regulator

[PolyX]

Linear quadratic regulator–1

PolyX

State-feedback:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= -Lx(t)\end{aligned}$$

Regulated variable:

$$z(t) = C_R x(t) + J_R u(t)$$

LQ regulator problem:

Find control function $u(t), t \geq 0$ which minimizes $\|z\|_2$ for every initial state $x(0)$.

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Linear quadratic regulator–2

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State-space solution:

(assumed:
 $J_R^T C_R = 0, J_R^T J_R = I$)

If (A, B) stabilizable, a unique optimal control exists

$$u(t) = -Lx(t), \quad L = G^T P$$

Where P is the least symmetric non-negative definite solution of:

$$PA + A^T P - PBB^T P + C_R^T C_R = 0$$

ARE

If (A, C_R) detectable, $\dot{x}(t) = (A - BL)x(t)$ is stable.

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Linear quadratic regulator-3

[PolyX]

Polynomial solution:

$$(sI - A)^{-1} B = N_R(s) D_R^{-1}(s)$$

right coprime

l2r conversion

spectral factorization

$$F^T(-s)F(s) = N_R^T(-s)C_R^T C_R N_R(s) + D_R^T(-s)D_R(s)$$

stable

$$X_L D_R(s) + Y_L N_R(s) = F(s)$$

constant solution

linear equation

$$L = X_L^{-1} Y_L$$

state feedback

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Linear Gaussian filter

[PolyX]

Linear Gaussian filter–1

PolyX

Linear system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_L u(t) \\ y(t) &= Cx(t) + J_L v(t)\end{aligned}$$

Gaussian white random process with zero mean and unity covariance:

$$v(t)$$

LQ filter problem:

Determine the conditional mean estimate $\hat{x}(t)$ of the current state $x(t)$ given the past observations $y(\tau), \tau \leq t$

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Linear Gaussian filter–2

PolyX

State-space solution: (for simplicity assumed:
 $G_L J_L^T = 0, J_L J_L^T = I$)

If (A, C) is detectable, the estimate is generated by the filter

$$\dot{\hat{x}}(t) = A\hat{x}(t) + QC^T[y(t) - C\hat{x}(t)]$$

Where P is the least symmetric non-negative definite solution of:

$$AQ + QA^T - QB^T BQ + B_L B_L^T = 0$$

ARE

The filter is obtained from the original system by applying an input injection

$$K = -QC^T$$

If (A, B_L) detectable, the filter is stable.

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Linear Gaussian filter-3

PolyX

Polynomial solution:

$$C(sI - A)^{-1} = D_L^{-1}(s) N_L(s)$$

left coprime

r2l conversion

spectral factorization

$$G(s)G^T(-s) = N_L(s)B_L B_L^T N_R^T(-s) + D_L(s)D_L^T(-s)$$

stable

$$A_L(s)X_R + B_L(s)Y_R = C(s)$$

constant solution

linear equation

$$K = Y_R X_R^{-1}$$

output injection

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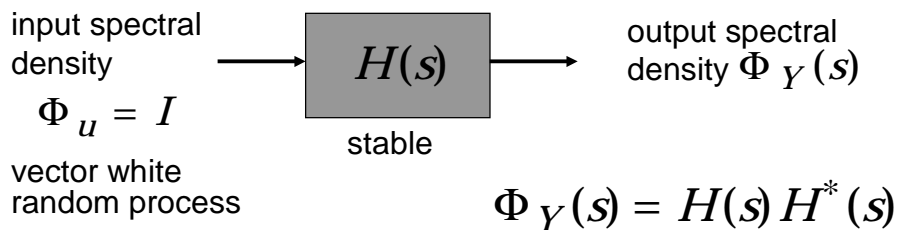
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Spectral factorization

PolyX

Spectral factorization - 1

PolyX



where

conjugation

$$H^*(s) = H^T(-s)$$

- para-Hermitian symmetric $\Phi_Y(s) = \Phi_Y^*(s)$
- positive definite on imaginary axis $\Phi_Y(j\omega) > 0 \quad \forall \omega \in \mathfrak{R}$

Spectral factorization - 2

PolyX

Conversed problem:

Given output spectral density, construct the (stable) system

Given rational matrix $\Phi_Y(s)$ such that

- para-Hermitian symmetric $\Phi_Y(s) = \Phi_Y^*(s)$
- positive definite on Im $\Phi_Y(j\omega) > 0 \quad \forall \omega \in \mathfrak{R}$

Find stable rational matrix $H(s)$ such that

Rational spectral factorization

$$\Phi_Y(s) = H(s) H^*(s)$$

Spectral factorization - 3

PolyX

Polynomial spectral factorization

Given polynomial matrix $M(s)$ that is

- para-Hermitian symmetric $M(s) = M^*(s)$
- positive (nonnegative) definite on Im

Find stable polynomial matrix $X(s)$ such that

$$M(s) = X(s) X^*(s)$$



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Spectral factorization - 4

PolyX

Example $a(s) = \underbrace{(2 - 2s + s^2)}_{1+j} \underbrace{(2 + s)}_{-2} \underbrace{(3 - s)}_3$

$$a^*(s) = a(-s) = \underbrace{(2 + 2s + s^2)}_{-1+j} \underbrace{(2 - s)}_2 \underbrace{(3 + s)}_{-3}$$

$$m(s) = a(s) a^*(s) \implies m(s) = x(s) x^*(s)$$

by inspection

$$x(s) = \underbrace{(2 + 2s + s^2)}_{-1+j} \underbrace{(2 + s)}_{-2} \underbrace{(3 + s)}_{-3}$$

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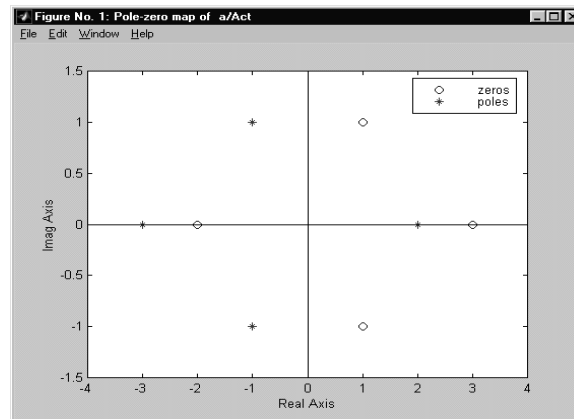
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Spectral factorization - 5

PolyX

Example

```
a = (2 - 2*s + s^2) * (2 + s) * (3 - s);  
pzplot(a, a')
```



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Spectral factorization - 6

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Canonical factorization:

$M(s)$ diagonally reduced

$X(s)$ row reduced

Uniqueness:

$$X'(s) = X(s)U$$

$$UU^T = I \quad \text{Constant, orthogonal}$$

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Spectral factorization - 6

[PolyX]

Polynomial J-spectral factorization

more
general

Given polynomial matrix $M(s)$ that is

- para-Hermitian symmetric $M(s) = M^*(s)$
- if zeros on Im , then of even multiplicity in each inv. pol.

Find stable polynomial matrix $X(s)$ such that

$$M(s) = X(s) J X^*(s)$$

$$J = \text{diag}\{+1, \dots, -1, \dots, 0, \dots\}$$

Signature
matrix

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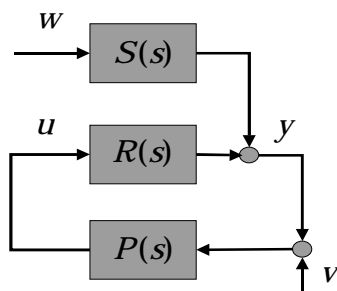
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LQG

[PolyX]

LQG - 1

PolyX



$$y(s) = R(s)u(s) + S(s)w(s)$$

$$u(s) = -P(s)(y(s) - v(s))$$

background noise $w(s) \dots Q_1$

observation noise $v(s) \dots R_1$

(independent, zero-mean,
stationary, white, Gaussian)

$$\lim_{t \rightarrow \infty} E(u^T(t) R_2 u(t) + y^T(t) Q_2 y(t))$$

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LQG - 2

PolyX

Polynomial solution

Given

- left coprime lmf
- coprime pmf's

$$A^{-1}[B \ C] = [R \ S]$$

$$A_0^{-1} B_0 = B_1 A_1^{-1} = R$$

Compute

- 2 spectral factorizations

$$AR_1 A^* + CQ_1 C^* = GG^*$$

noise

$$A_1^* R_2 A_1 + B_1^* Q_2 B_1 = F^* F$$

cost

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LQG - 3

PolyX

- 2 coprime pmf's

$$\begin{aligned} B_2 G_1^{-1} &= G^{-1} B \\ A_2 G_2^{-1} &= G^{-1} A \end{aligned}$$

- linear matrix polynomial equation (two-sided)

$$F^* [X \ Y] - Z^* [B_2 \ -A_2] = [A_1^* R_2 G_1 \ B_1^* Q_2 G_2]$$

- special solution (via division)

X_1, Y_1, Z_1 such that $F^{-1} Z_1$ strictly proper

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LQG - 4

PolyX

- coprime lmf

$$G_0^{-1} [X_0 \ Y_0] = [X_1 G_1^{-1} \ Y_1 G_2^{-1}]$$

- controller

$$P = X_0^{-1} Y_0$$

Solvability conditions

- stable greatest common left divisor of A, B
- (4 rational matrices strictly proper)
- nonsingular X_0

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LQG - 5

PolyX

$$F^* \begin{bmatrix} X_1 & Y_1 \end{bmatrix} - Z^* \begin{bmatrix} B_2 & -A_2 \end{bmatrix} = \begin{bmatrix} A_1^* R_2 G_1 & B_1^* Q_2 G_2 \end{bmatrix} \times \begin{bmatrix} G_1^{-1} A_1 \\ G_2^{-1} B_1 \end{bmatrix}$$

$$F^* \begin{bmatrix} X_1 G_1^{-1} A_1 & Y_1 G_2^{-1} B_1 \end{bmatrix} - Z^* G^{-1} \begin{bmatrix} B A_1 & -A B_1 \end{bmatrix} =$$

$$= A_1^* R_2 A_1 + B_1^* Q_2 B_1 = F^* F$$

$$X_1 G_1^{-1} A_1 + Y_1 G_2^{-1} B_1 = F$$

pole placement

$$X_0 A_1 + Y_0 B_1 = G_0 F$$

implied equation

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LQG - 6

PolyX

simpler solution for $C = B \quad A_0^{-1} B_0 = B_1 A_1^{-1} = R$

$$X_1 A_1 + Y_1 B_1 = F$$

Polynomial part

$$T_1 = \Pi(Y_1 A_0^{-1} G)$$

$$T_1 G_1^{-1} = \bar{G}^{-1} \bar{T}^{-1}$$

$$X_0 = \bar{G} X_1 + \bar{T}_1 B_0$$

$$Y_0 = \bar{G} Y_1 - \bar{T}_1 A_0$$

$$\implies X_0 A_1 + Y_0 B_1 = \bar{G} F$$

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Comparison with state-space solution



Comparison - 1

PolyX

$$J = E \left(\int_{-\infty}^{\infty} [x^T(t)Q_2x(t) + u^T(t)R_2u(t)] dt \right)$$

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$

$$y(t) = Cx(t) + v(t)$$

w, v Independent, white, Gaussian, zero means intensities $Q_1 \geq 0, R_1 > 0$

$Q_2 \geq 0, R_2 > 0$ weighting matrices

Comparison - 2

PolyX

State space solution - Separation theorem

Optimal control law $u(t) = -L_2 \hat{x}(t)$

$\hat{x}(t)$ is estimate of $x(t)$ generated by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L_1(t)(y(t) - C\hat{x}(t))$$

$$L_1 = P_1 C^T R_1^{-1} \quad L_2 = R_2^{-1} B^T P_2$$

$$AP_1 + P_1 A^T - P_1 C^T R_1^{-1} C^T P_1 + Q_1 = 0$$

$$P_2 A + A^T P_2 - P_2 B R_2^{-1} B^T P_2 + Q_2 = 0$$

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Comparison - 3

PolyX

(C, A)	detectable	} estimator exists and is stable
$\begin{bmatrix} -A^T & C^T R_1^{-1} C \\ Q_1 & A \end{bmatrix}$	no purely imaginary eigenvalue	

(A, B)	stabilizable	} regulator exists and is stable
$\begin{bmatrix} -A & B R_2^{-1} B^T \\ Q_2 & A^T \end{bmatrix}$	no purely imaginary eigenvalue	

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Comparison - 4

PolyX

If stabilizing R_1 exists but the pair (A, Q_1) is not stabilizable, then optimal estimator is not stable.

Take best stable estimator (R_1)

If stabilizing R_2 exists but the pair (Q_2, A) is not detectable, then optimal estimator is not stable.

Take best stable regulator(R_2)

What if R_1 or R_2 does not exist?

If no stable estimator and/or regulator?

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Comparison - 5

PolyX

Polynomial solution

$$(sI - A)^{-1} B = \frac{\bar{b}(s)}{a(s)}, \quad C(sI - A)^{-1} = \frac{\bar{c}(s)}{a(s)}$$

$$C(sI - A)^{-1} B = \frac{h(s)}{a(s)}$$

(A, B) controllable

(C, A) observable

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Comparison - 6

PolyX

$$a(s)R_1 a^*(s) + \bar{c}(s)Q_1 \bar{c}^*(s) = f(s)f^*(s)$$

$$a(s)R_2 a^*(s) + \bar{b}(s)Q_2 \bar{b}^*(s) = g^*(s)g(s)$$

$$a(s)x(s) + b(s)y(s) = f(s)g(s)$$

$$\deg y(s) < \deg a(s)$$

$$\frac{y(s)}{x(s)}$$

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Comparison - 7

PolyX

May work even if (Q_2, A) is undetectable and/or (A, Q_1) is unstabilizable and the corresponding eigenvalue is purely imaginary.

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Comparison - 7

PolyX

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 1]$

$$Q_1 = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_1 = 1, R_2 = 1$$

(A, Q_1) is not stabilizable and

$$\begin{bmatrix} -A^T & C^T R_1^{-1} C \\ Q_1 & A \end{bmatrix} \text{ has a zero eig.}$$



Stable estimator does not exist.

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Comparison - 8

PolyX

$$Q_1 = \begin{bmatrix} 16 & 0 \\ 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a(s) = s^2, b(s) = 1 + 4s + 6s^2$$

$$\bar{c}(s) = \begin{bmatrix} s^2 & s + 4s^2 & 1 + 4s + 6s^2 \end{bmatrix}, \bar{b}(s) = \begin{bmatrix} 1 \\ s \end{bmatrix}$$

$$f(s) = s(4 + s), g(s) = 4 + 3s + s^2$$

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Comparison - 9

[PolyX]

$$x(s) = s(7+s), \quad y(s) = 16s$$

$$\frac{y(s)}{x(s)} = -\frac{16}{7+s}$$

The controller has an unobservable mode. When minimally realize, it is the optimal controller.

The unobservable mode is not necessary for control and hence need **not be estimated !!!**

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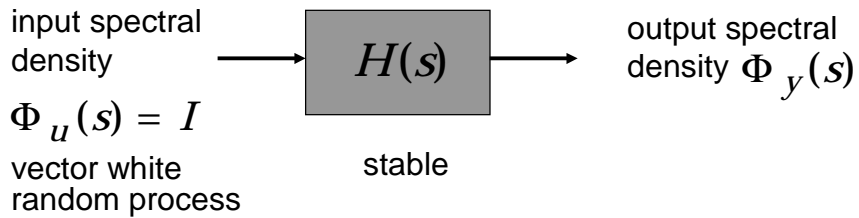
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H_2 Optimization

[PolyX]

H_2 Optimization- 1

PolyX



$$\Phi_Y(s) = H(s)H^*(s)$$

H_2 optimization = removing the stochastic interpretation from LQG

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H_2 Optimization- 2

PolyX

Variance of the output signal

$$\begin{aligned} E y^T(t) y(t) &= \text{tr} E y(t) y^T(t) \\ &= \text{tr} \int_{-\infty}^{\infty} H(j\omega) H^T(-j\omega) df \end{aligned}$$

2-norm $\|H\|_2$ of system transfer matrix

$$f = \frac{\omega}{2\pi}$$

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