

## Achievable feedback performance

**Introduction** Many of the properties of the MIMO feedback loop of Fig. 1 are characterized by its sensitivity matrix

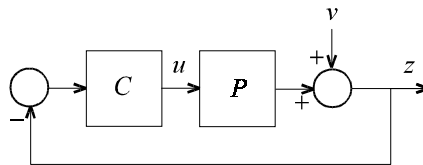
$$S = (I + PC)^{-1}$$

and complementary sensitivity matrix

$$T = (I + PC)^{-1} PC$$

These properties include the sensitivity of the feedback loop to disturbances and measurement noise, its response to reference inputs, and its stability and performance robustness (Kwakernaak, 1995).

$H_\infty$  optimization is a powerful tool for shaping the sensitivity functions so that all the design requirements are satisfied. In particular the mixed sensitivity problem is a useful design paradigm. The Polynomial Toolbox contains the routine `mixed_s` for solving SISO mixed sensitivity problems.



**Fig. 1. LTI feedback system**

Especially for complex design problems it is highly recommended to devote some time to *exploratory analysis* before attempting the actual design. This exploratory analysis serves to reveal the inherent possibilities and limitations of the control system. Part of this exploratory analysis always is the computation of the poles and zeros of the plant. Any right-half plane zeros and poles that are present impose essential constraints on the closed-loop bandwidth that may be achieved or is necessary (again see Kwakernaak, 1995, for a review).

For MIMO systems the open-loop pole and zero locations do not fully reveal the design limitations. For instance, in the SISO case the presence of a nearly cancelling pole-zero pair in the right-half plane predicts poor performance, with high peaks in

the sensitivity functions. The closer the pole and zero are, the higher the peak. In the case of MIMO systems the pole of such a pair may occur in a different “channel” than the zero so that the pole and zero do not interact and their adverse effects are not amplified such as in the SISO case.

To reveal the a priori design limitations more fully it may be useful to compute the minimal peak values of the  $\infty$ -norms of the sensitivity functions  $S$  and  $T$ . This is the purpose of this demo. Polynomial techniques lend themselves very well for the computation of these bounds and explicitly reveal their relation to the open-loop zero and pole locations. The peak values themselves are lower bounds for the peak values that are obtained for more realistic designs matched to the design specifications.

**SISO minimal peak values for the sensitivities**

For the SISO case the minimal peak values are derived in Kwakernaak (1985). We assume that the plant has the transfer function

$$P(s) = \frac{N(s)}{D(s)} = \frac{N_+(s)N_-(s)}{D_+(s)D_-(s)}$$

The polynomial  $N$  is the plant numerator polynomial and the polynomial  $D$  is the plant denominator polynomial.  $D_+$  and  $N_+$  are polynomials whose roots have strictly positive part and hence lie in the open right-half complex plane. The roots of the polynomials  $D_-$  and  $N_-$  all have nonpositive real part.

The minimal peak value of the  $\infty$ -norm of the sensitivity function  $S$  in the SISO case equals the minimal peak value of the magnitude of the sensitivity function  $|S(j\omega)|$ ,  $\omega \in \mathbb{R}$ . This minimal norm may be found by solving a special  $H_\infty$  optimization problem, namely that of minimizing  $\|S\|_\infty$ . The solution of this minimum sensitivity problem follows by solving the polynomial equation

$$\frac{1}{\gamma} D_+^* X_+^* = D_+ X_+ + N_+ Y_+$$

for the scalar  $\gamma$  and the polynomials  $X_+$  and  $Y_+$ . If  $D_+$  has degree  $d$  and  $N_+$  has degree  $n$  then  $X_+$  has degree  $n-1$  and  $Y_+$  has degree  $d-1$ . As we shall see this equation is equivalent to a generalized eigenvalue problem. The compensator that solves the minimum sensitivity problem has the transfer function

$$C_{\text{opt}}(s) = \frac{D_-(s)Y_+(s)}{N_-(s)X_+(s)}$$

If the plant is strictly proper then  $C_{\text{opt}}$  is nonproper. The optimal sensitivity function is given by

$$S_{\text{opt}}(s) = \gamma \frac{D_+(s) X_+(s)}{D_+^*(s) X_+^*(s)}$$

$S_{\text{opt}}$  has the so-called *equalizing property*, that is,  $|S_{\text{opt}}(j\omega)|$  is constant for all real  $\omega$  and equals  $|\gamma|$ . The minimal peak value hence is  $|\gamma|$ .

The properties of the minimum peak value may be summarized as follows:

If the plant has no right-half plane zeros then  $\|S_{\text{opt}}\|_{\infty} = 0$ , which is achieved by an infinite-gain compensator.

If the plant has no right-half plane poles but at least one right-half plane zero then  $\|S_{\text{opt}}\|_{\infty} = 1$ , which is obtained by taking  $C(s) = 0$ .

If the plant has at least one right-half plane zero and at least one right-half plane pole then  $\|S_{\text{opt}}\|_{\infty} > 1$ .

If the plant has a coinciding right-half plane pole-zero pair then  $\|S_{\text{opt}}\|_{\infty} = \infty$ .

For the minimal peak value of the complementary sensitivity function  $T$  similar results hold. The critical equation that needs to be solved now is

$$\frac{1}{\gamma} N_+^* Y_+^* = D_+ X_+ + N_+ Y_+$$

and the optimal complementary function is

$$S_{\text{opt}}(s) = \gamma \frac{N_+(s) Y_+(s)}{N_+^*(s) Y_+^*(s)}$$

This is the summary of the results for the minimization of  $\|T\|_{\infty}$ :

If the plant has no right-half plane poles then  $\|T_{\text{opt}}\|_{\infty} = 0$ , which is achieved by an infinite-gain compensator.

If the plant has no right-half plane zeros but at least one right-half plane pole then  $\|T_{\text{opt}}\|_{\infty} = 1$ , which is obtained by taking  $C(s) = 0$ .

If the plant has at least one right-half plane zero and at least one right-half plane pole then  $\|T_{\text{opt}}\|_{\infty} = \|S_{\text{opt}}\|_{\infty} > 1$ .

If the plant has a coinciding right-half plane pole-zero pair then  $\|T_{\text{opt}}\|_{\infty} = \infty$ .

Note that minimal sensitivity and minimal complementary sensitivity are not simultaneously achieved by the same compensator.

**SISO  
computation of  
the minimum  
peak value**

The only situation where some serious computation needs to be done once the open-loop poles and zeros are available arises when the plant has both right half plane zeros and poles. We then need to solve the polynomial equation

$$\frac{1}{\gamma} D_+^* X_+^* = D_+ X_+ + N_+ Y_+ \tag{1}$$

The solution of this problem provides the minimal peak value of both the sensitivity and the complementary sensitivity function. If  $D_+$  has degree  $d$  and  $N_+$  has degree  $n$  then  $X_+$  has degree  $n-1$  and  $Y_+$  has degree  $d-1$ . Write  $X_+$  and  $Y_+$  in terms of their coefficients as

$$X_+(s) = X_0 + X_1 s + \dots + X_{n-1} s^{n-1}, \quad Y_+(s) = Y_0 + Y_1 s + \dots + Y_{d-1} s^{d-1}$$

and introduce the column vectors

$$x = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{bmatrix}, \quad y = \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{d-1} \end{bmatrix}$$

Then it may be verified that the polynomial equation (1) we need to solve is equivalent to the matrix equation

$$\frac{1}{\gamma} S_{n+d-1}(D_+^*) J_n x = S_{n+d-1}(D_+) x + S_{n+d-1}(N_+) y \tag{2}$$

$S_m(A)$  denotes the (column) Sylvester resultant matrix of order  $m$  of the polynomial  $A$ , and  $J_n$  is the matrix

$$J_n = \text{diag}(1, -1, 1, -1, \dots, (-1)^{n-1})$$

To show that (2) amounts to a generalized eigenvalue problem we rearrange it in the form

$$\frac{1}{\gamma} \begin{bmatrix} S_{n+d-1}(D_+^*) J_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} S_{n+d-1}(D_+) & S_{n+d-1}(N_+) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is equivalent to the equation

$$Az = \lambda Bz \quad (3)$$

where

$$A = [S_{n+d-1}(D_+) \quad S_{n+d-1}(N_+)], \quad B = [S_{n+d-1}(D_+^*) J_n \quad 0]$$

$$z = \begin{bmatrix} x \\ y \\ u \end{bmatrix}, \quad \lambda = \frac{1}{\gamma}$$

We need the solution of (3) that corresponds to the real eigenvalue  $\lambda$  with the smallest size. The minimal peak sensitivity equals the inverse of the magnitude of the smallest eigenvalue.

### The macro minsens

We develop a new Polynomial Toolbox function minsens that computes the minimal peak value of the sensitivity functions. Its input arguments are the numerator polynomial  $N$  and the denominator polynomial  $D$  of the SISO plant.

As we develop the macro we test it for the plant with transfer function

$$P(s) = \frac{s^2 + 2s - 3}{s^2 - 4/9}$$

We input the data accordingly as

```
N = s^2+2*s-3;
D = s^2-4/9;
```

The first few lines of the m-file are

```
% minsens
% The function
%   p = minsens(N,D)
% computes the minimum peak value of the sensitivity and
% complementary sensitivity functions for the SISO plant
% with transfer function P = N/D that may be achieved by
% feedback
```

```
function p = minsens(N,D)
```

This provides the help text, and declares the function, its input arguments and its output arguments.

Normally a sequence of tests needs to follow this preamble to check whether  $N$  and  $D$  are really scalar polynomials but we dispense with this for the purpose of this demo.

Given the numerator and denominator polynomials we now compute their roots and use these to define the polynomials  $N_+$  and  $D_+$ :

```
% Compute the polynomials Nplus and Dplus whose roots are
% the roots of N and D, respectively, with positive real parts
rootsN = roots(N); rootsNplus = rootsN(find(real(rootsN)>0));
Nplus = mat2pol(poly(rootsNplus));
rootsD = roots(D); rootsDplus = rootsD(find(real(rootsD)>0));
Dplus = mat2pol(poly(rootsDplus));
keyboard
```

The MATLAB command `poly` is used to construct the polynomials  $N_{\text{plus}}$  and  $D_{\text{plus}}$  from their roots after which they are converted to Polynomial Toolbox format with the Toolbox command `mat2pol`. While developing the macro we end it with the `keyboard` command so that the results at that point may be inspected. In the present case typing the command

```
minsens(N,D)
```

results in the output

```
K>
```

Editing this to

```
K> Nplus, Dplus
```

and ending the line with a return results in the output

```
Nplus =
```

```

        -1 + s
Dplus =
        -0.67 + s

```

We now include two tests to see whether the peak value is either 0 or 1.

```

% Check whether p = 0 or p = 1
if isempty(Nplus)
    p = 0; return
elseif isempty(Dplus)
    p = 1; return
end

```

For the example that we are pursuing both tests fail so we are in the situation where the generalized eigenvalue problem needs to be solved. We first set it up.

```

% Solve the generalized eigenvalue problem
A = [ sylv(Dplus, 'col', n-1) sylv(Nplus, 'col', d-1) ];
J = 1;
for i = 2:n
    J(i,i) = -1*J(i-1,i-1);
end
B = [ sylv(Dplus, 'col', n-1)*J zeros(n+d,d) ];

```

Only one more line is needed to complete the macro:

```

p = 1/min(abs(eig(A,B)));

```

Calling

```

minsens(N,D)

```

results for the example at hand in

```
ans =
```

```
5.0000
```

We test a few more examples

No right-half plane zeros:

```
minsens(s+1,s-1)
```

```
ans =
```

```
0
```

One right-half plane zero but no right-half plane poles:

```
minsens(s-1,s+1)
```

```
ans =
```

```
1
```

Coinciding right-half plane pole-zero pair:

```
minsens(s-1,s-1)
```

```
Warning: Divide by zero.
```

```
ans =
```

```
Inf
```

### **MIMO minimal peak values for the sensitivities**

For the MIMO case no simple results to compute the minimal peak sensitivity such as for the SISO case are available. The only option seems to be to compute a full solution of the minimum sensitivity problem. The Polynomial Toolbox provides the routine `dssrch` for this purpose. It solves the standard  $H_\infty$  optimization problem, is numerically well-conditioned, fast, computes optimal solutions accurately and reliably, and can handle nonproper generalized plants as well as problems where the optimal compensator is nonproper. The routine requires the generalized plant of the standard problem to be in descriptor form. The Toolbox supplies the necessary conversion routines from polynomial matrix fraction form to descriptor form and vice-versa.

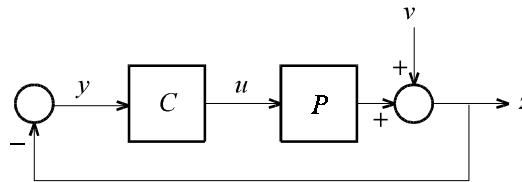


Fig. 2. Minimum sensitivity problem

To bring the minimum sensitivity problem into standard form we consider the block diagram of Fig. 2. When the loop is opened the signals are related as

$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} I & P \\ -C & P \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix}$$

This defines the generalized plant of the standard problem as

$$G = \begin{bmatrix} I & P \\ -C & -P \end{bmatrix}$$

If the plant  $P$  has the left coprime representation  $P = D^{-1}N$  then the generalized plant has the left coprime representation

$$G = \begin{bmatrix} D & 0 & P & 0 \\ -C & I & N & 0 \end{bmatrix}$$

After converting this left coprime fraction to descriptor form the routine `dssrch` may be called to solve the  $H_\infty$  optimization problem.

**MIMO example** We generate a random  $2 \times 2$  plant  $P = D^{-1}N$  by the commands

```
D = prand([1;2],2,2,'int')
```

```
D =
```

```
-8 + s    1 - 6s
```

```
6 + 6s    2 + s - s^2
```

```
N = prand([1;2],2,2,'int')
```

```
N =
```

$$\frac{-3 + 11s}{5} \frac{-1 + s}{-4 + s - 7s^2}$$

The zeros and poles of the plant are

```
Zeros = roots(N)
```

```
Zeros =
```

```
0.0379 + 0.8051i
```

```
0.0379 - 0.8051i
```

```
0.3399
```

```
Poles = roots(D)
```

```
Poles =
```

```
45.5167
```

```
-1.0000
```

```
0.4833
```

Inspection shows that the plant has both right-half plane poles and zeros so by analogy to the SISO problem we expect a minimum peak sensitivity of at least 1.

We first convert the generalized plant into descriptor form:

```
Dg = [D zeros(2,2); eye(2,2) eye(2,2)];
```

```
Ng = [D N; zeros(2,2) zeros(2,2)];
```

```
[A,B,C,D,E] = lmf2dss(Ng,Dg)
```

```
A =
```

```
8 -47 0
```

```
-6 37 1
```

```
-6 38 0
```

```
B =
```

```
0 0 85.0000 336.0000
0 0 -66.0000 -264.0000
0 0 -61.0000 -276.0000
```

C =

```
1 -6 0
0 -1 0
-1 6 0
0 1 0
```

D =

```
1.0000 0 11.0000 43.0000
0 1.0000 0 7.0000
-1.0000 0 -11.0000 -43.0000
0 -1.0000 0 -7.0000
```

E =

```
1 0 0
0 1 0
0 0 1
```

The solution of the  $H_\infty$  problem is initiated by typing

```
[Ak,Bk,Ck,Dk,Ek,gopt,clpoles] = dssrch(A,B,C,D,E,2,2,0.5,5)
```

The resulting output is

Ak =

```
0.5209 1.0787
-0.9706 -0.1370
```

Bk =

-0.9241 -9.2055

1.7746 -0.6237